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Properties of Topological Spaces

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Abstract:
In this paper, the definitions and examples of topological spaces are expressed. Some topological spaces; namely, T_1-spaces, Hausdorff spaces, regular spaces and normal spaces are discussed with some examples. The theorems which exhibit alternate of defining a topology on a set, using as primitives the neighborhood of a point and closure of a set are stated. Very simple characterizations of topological spaces are studied.

Keywords **—Hausdorff spaces, normal spaces, open set, regular spaces, T_1-spaces.**

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I. INTRODUCTION

General topology, also called point set topology, has recently become an essential part of the mathematical background of both graduate and under graduate students. Each article begins with clear statements of pertinent definition, principles and theorem together with illustrative and other descriptive material. This is followed by graded sets of solved and problems[2].

II. DEFINITIONSAND EXAMPLES OF TOPOLOGICAL SPACES

Basic definitions and examples are given[1].

A. Definition

Let X be a non-empty set. A class $\tau$of subsets of X is a *topology*on X iffτ satisfies the following axioms.

$[O_1]$ X and $\varnothing$ belong to τ.

$[O_2]$ The union of any number of sets in τ belongs to τ.

$[O_3]$ The intersection of any two sets in τ belongs to τ.

The members of τ are then called *τ-open sets*, or simply*open sets*, and X together with τ, that is, the pair (X, τ) is called a *topological space*.

B. Example

Let u denote the class of all open sets of real numbers. Then u is a usual topology on P. Similarly, the class u of all open sets in the plane P^2 is a usual topology on P^2.

C .Example

Consider the following classes of subsets of X = {a, b, c, d, e}.

$\tau_1 = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$,

$\tau_2 = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$,

$\tau_3 = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{a, b, d, e\}\}$.

Observe that τ_1 is a topology on X since it satisfies the three axioms $[O_1]$, $[O_2]$ and $[O_3]$.

But τ_2 is not a topology on X since the union

$$\{a, c, d\} \cup \{b, c, d\} = \{a, b, c, d\}$$

of two members of τ_2 does not belong to τ_2, that is, τ_2 does not satisfy the axiom $[O_2]$.

Also, τ_3 is not a topology on X since the intersection

$$\{a, c, d\} \cap \{a, b, d, e\} = \{a, d\}$$

of two sets in τ_3 does not belong to τ_3, that is, τ_3 does not satisfy the axiom $[O_3]$.

D. Definitions

Let Δ denote the class of all subsets of X. Then Δ satisfies the axioms for a topology on X. This topology is called the *discrete topology*; and X together with its discrete topology, that is, the pair (X, Δ), is called a *discrete topological space*or simply a*discrete space.*

E. Definitions

The class g = {X, $\varnothing$}, consisting of X and $\varnothing$ alone, is itself a topology on X. It is called the *indiscrete topology* and X together with its indiscrete topology, that is, (X, g) is called an *indiscrete topological space* or simply an *indiscrete space.*

E. Definition

Let τ denote the class of all subsets of X whose complements are finite together with the empty set $\varnothing$. This class τ is also a topology on X. It is called the *cofinite topology* or the *T_1-topology* on X.

F. Example

Each of the classes

$$\tau_1 = \{X, \varnothing, \{a\}\} \text{ and } \tau_2 = \{X, \varnothing, \{b\}\}$$

is a topology on X = {a, b, c}. But the union

$$\tau_1 \cup \tau_2 = \{X, \varnothing, \{a\}, \{b\}\}$$

is not a topology on X since it violates $[O_2]$.

That is, $\{a\} \in \tau_1 \cup \tau_2$, $\{b\} \in \tau_1 \cup \tau_2$ but $\{a\} \cup \{b\} = \{a, b\}$ does not belong to$\tau_1 \cup \tau_2$.

G. Definitions

Let X be topological space. A point $p \in X$ is an *accumulation point or limit point* of a subset A of X iff every open set G containing p, contains a point of A different from p, that is,

G open, $p \in G$ implies $(G \setminus \{p\}) \cap A \neq \varnothing$. The set of accumulation point of A, denoted by A', is called the *derived set*of A.

H. Definition

Let X be a topological space. A subset A of X is a *closed set* iff its complement A^c is an open set.

I. Example

The class τ= {X, $\varnothing$,{a},{c, d},{a, c, d},{b, c, d, e}}defines a topology on X = {a, b, c, d, e}.The closed subsets of X are $\varnothing$, X, {b, c, d, e}, {a, b, e}, {b, e}, {a}, that is, the complements of the open subsets of X. Note that there are subsets of X, such as {b, c, d, e} which are both open and closed and there are subsets of X, such as {a, b} which are neither open nor closed.

J. Theorem

A subset A of a topological space X is closed if and only if A contains each of its accumulation points.

In other words, a set A is closed if and only if the derived set A' of A is a subset of A, that is, $A' \subset A$.

Proof:

Suppose A is closed and let $p \notin A$, that is $p \in A^c$. But A^c, the complement of a closed set, is open; hence $p \notin A'$ for A^c is an open set such that

$$p \in A^c \quad \text{and} \quad A^c \cap A = \varnothing.$$

Thus $A' \subset A$ if A is closed.

Now assume $A' \subset A$; we show that A^c is open.

Let $p \in A^c$; then $p \notin A'$,

so there exists an open set G such that

$$p \in G \quad \text{and} \quad (G \setminus \{p\}) \cap A = \varnothing.$$

But $p \notin A$; hence $G \cap A = (G \setminus \{p\}) \cap A = \varnothing$.

So $G \subset A^c$. Thus p is an interior point of A^c and so A^c is open.

K. Definition

Let A be a subset of a topological space X. The *closure* of A, denoted by $\bar{A}$ is the intersection of all closed supersets of A.

In other words, if $\{F_i : i \in I\}$ is the class of all closed subsets of X containing A, then

$$\bar{A} = \cap_i F_i.$$

Observe that $\bar{A}$ is a closed set since it is the intersection of closed sets[4]. Furthermore, $\bar{A}$ is the smallest closed superset of A, that is, if F is a closed set containing A, then

$$A \subset \bar{A} \subset F.$$

Accordingly, a set A is closed if and only if $A = \bar{A}$.

L. Example

Consider the topology τ on X = {a, b, c, d, e} is {X, ∅, {a}, {c, d}, {a, c, d}, {b, c, d, e}}.The closed subsets of X are ∅, X, {b, c, d, e}, {a, b, e}, {b, e}, {a}.

Accordingly,

$$\overline{\{b\}} = \{b,e\}, \quad \overline{\{a,c\}} = X, \quad \overline{\{b,d\}} = \{b,c,d,e\}.$$

M. Definition

Let τ_1 and τ_2 be topologies on a non-empty set X. Suppose that each τ_1-open subset of X is also a τ_2-open subset of X. That is, suppose that τ_1 is a subclass of τ_2.Then we say that τ_1 is *coarser* or *smaller* than τ_2 or that τ_2 is *finer* or *larger* than τ_1.

N.Definitions

Let A be a non-empty subset of a topological space (X, τ). The class τ_A of all intersections of A with τ-open subsets of X is a topology on A; it is called the *relative topology* on A or the *relativization* of τ to A, and the topological space (A, τ_A) is called a *subspace* of (X, τ).In other words, a subset H of A is a τ_A-open set, that is, open relative to A, if and only if there exists a τ-open subset G of X such that

$$H = G \cap A.$$

O. Example

Consider the topology

$$\tau = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$$

on X = {a, b, c, d, e}, and the subset A = {a, d, e} of X.

Observe that

$$X \cap A = A, \quad \{a\} \cap A = \{a\}, \quad \{a,c,d\} \cap A = \{a,d\},$$
$$\varnothing \cap A = \varnothing, \quad \{c,d\} \cap A = \{d\}, \quad \{b,c,d,e\} \cap A = \{d,e\}.$$

Hence the relative topology on A is

$$\tau_A = \{A, \varnothing, \{a\}, \{d\}, \{a, d\}, \{d, e\}\}.$$

II. T_1-SPACES

A. Definition

A topological space X is a *T_1-space* iff it satisfies the following axiom[2]:

[T_1] Given any pair of distinct points $a, b \in X$, each belongs to an open set which does not contain the other.

In other words, there exists open sets G and H such that

$$a \in G, \quad b \notin G \quad \text{and} \quad b \in H, \quad a \notin H.$$

The open sets G and H are not necessarily disjoint.

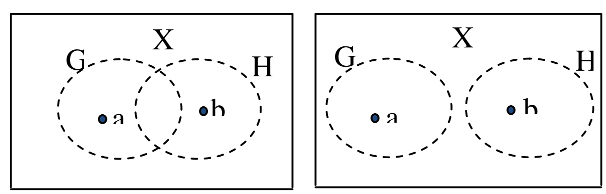

Fig.1 T_1 –Spaces

B. Theorem

A topological space X is a T_1-space if and only if every singleton subset {p} of X is close.

Proof:

Suppose X is a T_1-space and $p \in X$. We have to show that $\{p\}^c$ is open.

Let $x \in \{p\}^c$.

Then $x \neq p$, and so by [T_1], there exists an open set G_x such that $x \in G_x$ but $p \notin G_x$.

Hence $x \in G_x \subset \{p\}^c$, and hence

$$\{p\}^c = \cup\{G_x : x \in \{p\}^c\}.$$

Accordingly $\{p\}^c$, a union of open sets, is open and $\{p\}$ is closed.Conversely, suppose $\{p\}$ is closed for every $p \in X$.

Let $a, b \in X$ with $a \neq b$. Now, $a \neq b \Rightarrow b \in \{a\}^c$;

Hence $\{a\}^c$ is an open set containing b but not containing a.

Similarly $\{b\}^c$ is an open set containing a but not containing b. Accordingly, X is a T_1-space.

C. Example

Every metric space X is a T_1-space since we proved that finite subsets of X are closed.

D. Example

Consider the topology $\tau = \{X, \varnothing, \{a\}\}$on the set X = {a, b}. Observe that X is the only open set containing b, but it also contains a. Hence (X, τ) does not satisfy [T_1], that is , (X, τ) is not a T_1-space. Note that the singleton set {a} is not closed since its complement $\{a\}^c = \{b\}$ is not open.

E. Example

Let X be a T_1-space. Then the following are equivalent;

(i) $p \in X$ is an accumulation point of A.

(ii) Every open set containing p contains an infinite number of points of A.

For, if by the definition of an accumulation point of a set, (ii) $\Rightarrow$ (i).

Let $p \in X$ is an accumulation point of A.

Suppose G is an open set containing p and only containing a finite number of point of A different from p;

say $B = (G \setminus \{p\}) \cap A = \{a_1, a_2, ..., a_n\}$.

Now B, a finite subset of a T_1-space, is closed and so B^c is open. Set $H = G \cap B^c$. Then H is open, $p \in H$ and H contains no points of A different from p.

Hence p is not an accumulation point of A and so this contradicts the fact that p is an accumulation point of A.

Thus every open set containing p contains an infinite number of points of A.

III.HAUSDORFF SPACES

A. Definition

A topological space X is a *Hausdorffspace* or *T_2-space*iff it satisfies the following axiom[2]:

[T_2] Each pair of distinct points $a, b \in X$ belong respectively to disjoint open sets.

In other words, there exist open sets G and H such that

$$a \in G, b \in H \quad \text{and} \quad G \cap H = \varnothing.$$

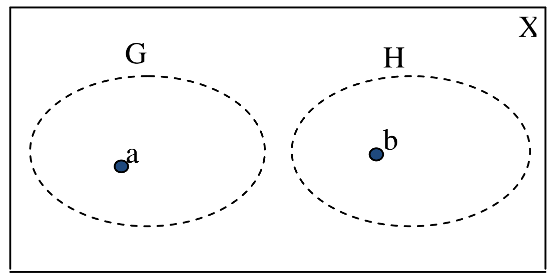

Fig.2 T_2 –Spaces (Hausdorff Space)

B. Theorem

Every metric space is a Hausdorff space.

Proof:

Let X be a metric space and $a, b \in X$ be distinct points; hence by the definition of metric [M_4], d(a, b) = ε> 0.

Consider the open spheres

$G = S(a, \frac{1}{3}\varepsilon)$ and $H = S(b, \frac{1}{3}\varepsilon)$, centered at a and b respectively. We claim that G and H are disjoint.

For if $p \in G \cap H$,then $d(a,p) < \frac{1}{3}\varepsilon$ and $d(p,b) < \frac{1}{3}\varepsilon$; hence by the Triangle Inequality,

$$d(a,p) \le d(a,p) + d(p,b) < \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \frac{2}{3}\varepsilon.$$

But this contradicts the fact that d(a, b) = ε.

Hence G and H are disjoint, that is, a and b belong respectively to the disjoint open spheres G and H.

Accordingly, X is Hausdorff.

C. Example

Let τ be the cofinite topology on the real line P. We can show that (P, τ) is not Hausdorff. Let G and H be any non-empty τ-open sets[3].

Now G and H are infinite since they are complements of finite sets. If $G \cap H = \varnothing$, then G, an infinite set, would be contained in the finite complement of H; hence G and H are not disjoint. Accordingly, no pair of distinct points in ¡ belongs, respectively, to disjoint τ-open sets. Thus, T_1-spaces need not be Hausdorff.

D. Theorem

If X is a Hausdorff space, then every convergent sequence in X has a unique limit.

Proof:

Suppose $(a_1, a_2, ...)$ converges to a and b and suppose $a \neq b$.

Since X is a Hausdorff, there exist open sets G and H such that $a \in G, \quad b \in H \quad \text{and} \quad G \cap H = \varnothing$.

By hypothesis, (a_n) converges to a;

Hence $\exists n_0 \in N$ such that $n > n_0$ implies $a_n \in G$,

that is, G contains all except a finite number of the terms of the sequence. But G and H are disjoint; hence H can only contain those terms of the sequence which do not belong to G and there are only a finite number of these.

Accordingly, (a_n) cannot converge to b.

But this violates the hypothesis; hence a = b.

E. Definition

A topological space X is called a *first countable space* if it satisfies the following axiom, called the *first axiom of countability***.**

$[C_1]$ For each point $p \in X$ there exists a countable class B_p of open sets containing p such that every open set G containing p also contains a member of B_p.

F. Theorem

Let X be first countable. Then X is Hausdorff if and only if every convergent sequence has a unique limit.

Proof:

By the above theorem, if X is Hausdorff, then every convergent sequence has a unique limit.

Conversely suppose that every convergent sequence has a unique limit. Assume that X is not Hausdorff. Then $\exists\ a, b \in X, \quad a \neq b$, with the property that every open set containing a has a non-empty intersection with every open set containing b.

Now let $\{G_n\}$ and $\{H_n\}$ be nested local bases at a and b respectively.

Then $G_n \cap H_n \neq \varnothing$ for every $n \in N$, and so there exists $(a_1, a_2, ...)$ such that $a_1 \in G_1 \cap H_1, \quad a_2 \in G_2 \cap H_2, \cdots$.

Accordingly, (a_n) converges to both a and b.

This contradicts the fact that every convergent sequence has a unique limit.

Hence X is a Hausdorff space.

IV. REGULAR SRPACES

A. Definition

A topological space X is *regular*iff it satisfies the following axiom[2]:

[R] If F is a closed subset of X and $p \in X$ does not belong to F, then there exists disjoint open sets G and H such that $F \subset G \quad \text{and} \quad p \in H$.

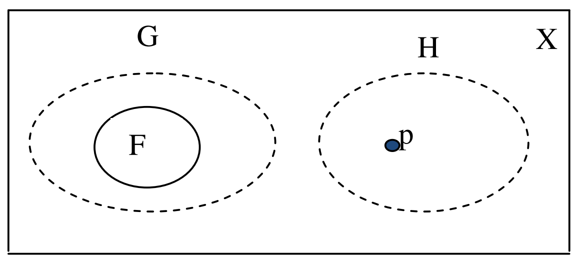

Fig.3 Regular Spaces

B. Example

Consider the topology τ= {X, ∅, {a}, {b, c}} on the set X = {a, b, c}. Observe that the closed subsets of X are also X, ∅, {a} and {b, c} and that (X, τ) does satisfy [R].

On the other hand, (X, τ) is not a T_1-space since there are finite sets, for example {b}, which are not closed.

C. Definition

A regular space X which also satisfies the separation axiom $[T_1]$, that is, a regular T_1-space is called a *T_3-space*.

D. Example

Let X be a T_3-space. Then we can show that X is also a Hausdorff space, that is, a T_2-space.

Let $a, b \in X$ be distinct points. Since X is a T_1-space, $\{a\}$ is a closed set; and since a and b are distinct, $b \notin \{a\}$.

Accordingly, by [R], there exist disjoint open sets G and H such that $\{a\} \subset G$ and $b \in H$.

Hence a and b belong respectively to disjoint open sets G and H.

V.NORMAL SPACES

A. Definition

A topological space X is *normal*iff X satisfies the following axiom:

[N] If F_1 and F_2 are disjoint closed subsets of X, then there exist disjoint open sets G and H such that $F_1 \subset G$ and $F_2 \subset H$.

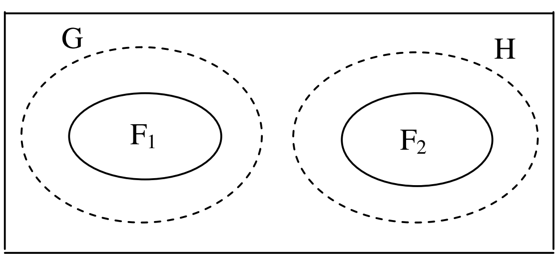

Fig.4Normal Spaces

B. Theorem

A topological space X is normal if and only if for every closed set F and open set H containing F there exists an open set G such that

$$F \subset G \subset \bar{G} \subset H.$$

Proof:

Suppose X is normal. Let $F \subset H$, with F closed and H open. Then H^c is closed and $F \cap H^c = \varnothing$. But X is normal; hence there exists open sets G, G^* such that $F \subset G$, $H^c \subset G^*$ and $G \cap G^* = \varnothing$.

But $G \cap G^* = \varnothing \Rightarrow G \subset G^{*c}$ and $H^c \subset G^* \Rightarrow G^{*c} \subset H$.

Furthermore, G^{*c} is closed; hence $F \subset G \subset \bar{G} \subset G^{*c} \subset H$.

Conversely suppose that for every closed set F and open set H containing F there exists an open set G such that $F \subset G \subset \bar{G} \subset H$.

Let F_1 and F_2 be disjoint closed sets.

Then $F_1 \subset F_2^c$ and F_2^c is open.

By hypothesis, there exists an open set G such that $F_1 \subset G \subset \bar{G} \subset F_2^c$.

But $\bar{G} \subset F_2^c \Rightarrow F_2 \subset \bar{G}^c$ and $G \subset \bar{G} \Rightarrow G \cap \bar{G}^c = \varnothing$.

Furthermore, $\bar{G}^c$ is open.

Thus $F_1 \subset G$ and $F_2 \subset \bar{G}^c$ with G, $\bar{G}^c$ disjoint open sets; hence X is normal.

C. Example

Consider the topology τ= {X, ∅, {a}, {b}, {a, b}} on the set X = {a, b, c}. Observe that the closed sets are X, ∅, {b, c}, {a, c} and {c}.

If F_1 and F_2 are disjoint closed subsets of (X, τ), then one of them, say must be the empty set ∅.

Hence ∅ and X are disjoint open sets and $F_1 \subset \varnothing$ and $F_2 \subset X$. In other words, (X, τ) is a normal space.

On the other hand, (X, τ) is not a T_1-space since the singleton set {a} is not closed. Furthermore, (X, τ) is not a regular space since $a \notin \{c\}$ and the only open superset of the closed set {c} is X which also contains a.

D. Definition

A normal space X which also satisfies the separation axiom [T_1], that is, a normal T_1-space, is called a ***T_4-space*.**

E. Example

Let X be a T_4-space. Then X is also a regular T_1-space, that is, T_3-space.

For suppose F is a closed subset of X and $p \in X$ does not belong to F. By [T_1], {p} is closed; and since F and {p} are disjoint, by [N], there exist disjoint open sets G and H such that $F \subset G$ and $p \in \{p\} \subset H$.

The following diagram illustrates the relationship between the spaces discussed in this paper.

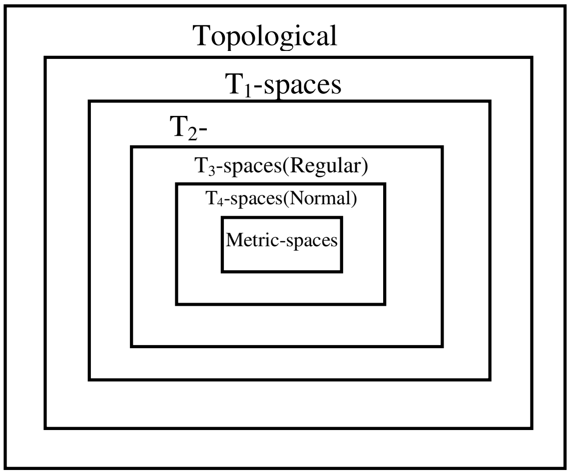

Fig.5The Relationship between the Topological Spaces

VI. CONCLUSIONS

In the discussion of properties of topology spaces, some topological spaces with illustrated examples are described. Many properties of a topological space X depend upon the distribution of the open sets in the space. A space is more likely to be separable or first or second countable, if there are few open sets, on the other hand, an arbitrary function on X to some topological space is more likely to be continuous or a sequence to have a unique limit, if the space has many open sets. The solved problems serve to illustrate and amplify the theory, bring into sharp focus those fine points without which the student continually feels himself an unsafe ground and provide the repetition of basic principles so vital to effective learning. The relationship between the topological spaces is illustrated.

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