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Closed –Form Solulion to Motion of A Ballistic Missile Launched At A Certain Angle of Inclination to The Horizon

SN MAITRA

Retired Head of Mathematics Department

National Defence Academy

Pune,India

ABSTRACT

In this paper relation between velocity and inclination of the missile is determined without resorting to any approximation. Atmospheric drag and gravitational field are duly taken into consideration. The horizontal distance described and altitude attained by the missile and the time taken are obtained as non-closed-form integrals with respect to the trajectory inclination. However, since because of high velocity of the missile reduction of its path inclination from its initial launch angle is small, an attempt has been made to approximately evaluate these integrals without loss of generality. Thus a complete solution to the missile performance is worked out.

INTRODUCTION

MS.Ganagi[1] and his co-author derived an approximate closed-form solution for trajectory of a ballistic projectile neglecting variation of inclination of the trajectory in a small interval of time. They have determined the trajectory of the missile using a suitable time step vis-a-vis by joining the positions of the projectile evaluated after every time step and that way developed twelve steps. Angelo Miele[2] gave some approximate partial closed-form solutions for non-lifting short range missile.

In this design is considered non-lifting ballistic missile launched at an angle of inclination to the horizon in atmosphere and gravitational field so as to obtain analytical/ closed-form solutions to its motion. The atmospheric[2] drag is proportional to the square of the velocity.The lift is very small and is neglected.If X and h denote respectively horizontal distance described and height attained at time t with velocity V at an angle γ of inclinationto the horizontal,

$$\frac{dX}{dt} = V\cos\gamma, \qquad \frac{dh}{dt} = V\sin\gamma \qquad (1)$$

In conformity with Angelo Miele's[2] analysis the equations of motion of the ballistic missile in the tangent and normal directions are written as

$$CV^2 + \sin\gamma + \frac{1}{g}\frac{dV}{dt} = 0 \qquad (2)$$

$$\frac{V}{g}\frac{d\gamma}{dt} + \cos\gamma = L = 0 \qquad (3)$$

where C is the constant of proportionality and g the acceleration due to gravity.

VELOCITY OF THE MISSILE

Combining equations (1) ,(3)and (2) and eliminating t, one gets

$$\cos\gamma\frac{dV}{d\gamma} - V\sin\gamma = CV^3$$

Or, $\frac{d(V\cos\gamma)}{d\gamma} = CV^3\frac{\cos^3\gamma}{\cos^3\gamma}$

$$\frac{1}{V^3\cos^3\gamma}\frac{d(V\cos\gamma)}{d\gamma} = C\sec^3\gamma \qquad (4)$$

Integrating which and utilizing the initial conditions:

At time t=0, $\gamma = \gamma_0, V = V_0, X = 0$ and $h = 0$ (5)

$$\frac{1}{V^2\cos^2\gamma} = C[\{\sec\gamma_0\tan\gamma_0 + \log(\sec\gamma_0 + \tan\gamma_0)\} - \{\sec\gamma\tan\gamma + \log(\sec\gamma + \tan\gamma)\}] + \frac{1}{V_0^2\cos^2\gamma_0} \qquad (6)$$

whereas $\int \sec^3\gamma d\gamma$

$= \int \sec\gamma d(\tan\gamma)$

$=\sec\gamma\tan\gamma - \int \sec\gamma(\sec^2\gamma - 1)d\gamma = \sec\gamma\tan\gamma - \sec^3\gamma d\gamma + \int \sec\gamma d\gamma$

$0r,2\int \sec^3\gamma d\gamma = \sec\gamma\tan\gamma + \log(\sec\gamma + \tan\gamma)$

$I_3=\frac{C}{2}\int_{\gamma}^{\gamma_0} \sec^3\gamma d\gamma = \frac{C}{2}\{\sec\gamma_0\tan\gamma_0 + \log(\sec\gamma_0 + \tan\gamma_0) - \sec\gamma\tan\gamma - \log(\sec\gamma + \tan\gamma)\}$ (7)

which is evaluated and formulated in almost all textbooks of Integral Calculus.Equation(6) gives a relation between the velocity and the path inclination without adhering to any approximation. When the projectile travels horizontally, the velocity V_s is given by putting $\gamma = 0$ in (6)

$$\frac{1}{V_s{}^2} = C[\{\sec\gamma_0\tan\gamma_0 + \log(\sec\gamma_0 + \tan\gamma_0)\}] + \frac{1}{V_0^2\cos^2\gamma_0} \quad (8)$$

The expression for velocity from (6) is rewritten as

$$V^2 = \frac{\sec^2\gamma}{C[\{\sec\gamma_0\tan\gamma_0+\log(\sec\gamma_0+\tan\gamma_0)\}-\{\sec\gamma\tan\gamma+\log(\sec\gamma+\tan\gamma)\}]+\frac{1}{V_0^2\cos^2\gamma_0}} \quad (9)$$

HORIZONTAL DISTANCE AND ALTITUDE

Combining (1),((3) and (9) in order to eliminate V and t, the horizontal distance X and altitude h can be obtained as integrals with respect to γ that cannot be evaluated in closed form:

$$X=\frac{1}{Cg}\int_{\gamma}^{\gamma_0} \frac{\sec^2\gamma d\gamma}{[\{\sec\gamma_0\tan\gamma_0+\log(\sec\gamma_0+\tan\gamma_0)\}-\{\sec\gamma\tan\gamma+\log(\sec\gamma+\tan\gamma)\}]+\frac{1}{CV_0^2\cos^2\gamma_0}} \quad (10)$$

$$h=\frac{1}{Cg}\int_{\gamma}^{\gamma_0} \frac{\sin\gamma\sec^3\gamma d\gamma}{[\{\sec\gamma_0\tan\gamma_0+\log(\sec\gamma_0+\tan\gamma_0)\}-\{\sec\gamma\tan\gamma+\log(\sec\gamma+\tan\gamma)\}]+\frac{1}{CV_0^2\cos^2\gamma_0}} \quad (11)$$

These two definite integrals can be evaluated numerically by Simpson's rule or by some other method of approximation.The missile is launched at an angle of elevation γ_0>flight-pathinclination γ at all time instants because of gravitational pull.Then at the sight of equations (10) and(11),

$\{\sec\gamma_0\tan\gamma_0 + \log(\sec\gamma_0 + \tan\gamma_0)\} + \frac{1}{V_0^2\cos^2\gamma_0} > \{\sec\gamma\tan\gamma + \log(\sec\gamma + \tan\gamma)\}$ (12)

which suggests that an infinite Binomial expansion can be utilized in equations (10) and (11)and as such can be approximated neglecting the square and other higher powers of the relatively smaller terms to facilitate solutions/integrations in closed form without sacrificing the accuracy in reaching the destination or striking the specified target in a short interval of time:

$$X = \frac{1}{Cg\{\sec\gamma_0\tan\gamma_0 + \frac{1}{CV_0^2\cos^2\gamma_0} + \log(\sec\gamma_0 + \tan\gamma_0)\}^2}\int_{\gamma}^{\gamma_0}[\{\sec\gamma_0\tan\gamma_0 + \log(\sec\gamma_0 + \tan\gamma_0)\} + \{\sec\gamma\tan\gamma + \log(\sec\gamma + \tan\gamma)\} + \frac{1}{CV_0^2\cos^2\gamma_0}]\sec^2\gamma d\gamma$$

$[d(\sec\gamma) = \sec\gamma\tan\gamma d\gamma]$

$$= \frac{1}{Cg\{\sec\gamma_0\tan\gamma_0 + \frac{1}{CV_0^2\cos^2\gamma_0} + \log(\sec\gamma_0 + \tan\gamma_0)\}^2}\left[\left\{\sec\gamma_0\tan\gamma_0 + \log(\sec\gamma_0 + \tan\gamma_0) + \frac{1}{CV_0^2\cos^2\gamma_0}\right\}(\tan\gamma_0 - \tan\gamma) + \frac{\sec^3\gamma_0 - \sec^3\gamma}{3} + (\tan\gamma_0)\log(\sec\gamma_0 + \tan\gamma_0) - (\tan\gamma)\log(\sec\gamma + \tan\gamma)\} - \sec\gamma_0 + \sec\gamma\right] \quad (13)$$

$$h = \int_{\gamma}^{\gamma_0}\frac{1}{Cg\{\sec\gamma_0\tan\gamma_0 + \frac{1}{CV_0^2\cos^2\gamma_0} + \log(\sec\gamma_0 + \tan\gamma_0)\}^2}[\{\sec\gamma_0\tan\gamma_0 + \log(\sec\gamma_0 + \tan\gamma_0)\} - \{\sec\gamma\tan\gamma + \log(\sec\gamma + \tan\gamma)\} + \frac{1}{CV_0^2\cos^2\gamma_0}]\tan\gamma\sec^2\gamma d\gamma$$

$$h = \frac{1}{Cg\{\sec\gamma_0\tan\gamma_0 + \frac{1}{CV_0^2\cos^2\gamma_0} + \log(\sec\gamma_0 + \tan\gamma_0)\}^2}\left[\left\{\sec\gamma_0\tan\gamma_0 + \log(\sec\gamma_0 + \tan\gamma_0) + \frac{1}{CV_0^2\cos^2\gamma_0}\right\}\frac{(\tan^2\gamma_0 - \tan^2\gamma)}{2} - j\right] \quad (14)$$

where $J = \int_{\gamma}^{\gamma_0}\{\sec\gamma\tan\gamma + \log(\sec\gamma + \tan\gamma)\}\sec^2\gamma\tan\gamma d\gamma$

$= \int_{\gamma}^{\gamma_0}\{sec(^5\gamma - sec^3\gamma)d\gamma + \frac{1}{2}\log(\sec\gamma + \tan\gamma)\}\, d(\tan^2\gamma)\}$ (Integrating by parts) vide relation(7)

$j = I_5 - I_3 + \frac{1}{2}[-\{\log(\sec\gamma + \tan\gamma)\}(\tan^2\gamma) + \{\log(\sec\gamma_0 + \tan\gamma_0)\}(\tan^2\gamma_0) - \int_{\gamma}^{\gamma_0} \sec\gamma\,(\sec^2\gamma - 1)]d\gamma$

$j = I_5 - 2I_3 + \frac{1}{2}[-\{\log(\sec\gamma + \tan\gamma)\}(\tan^2\gamma) + \{\log(\sec\gamma_0 + \tan\gamma_0)\}(\tan^2\gamma_0) + \sec\gamma\tan\gamma - \sec\gamma_0\tan\gamma_0]$ (15)

Where $I_5 = \int_{\gamma}^{\gamma_0} sec^5\gamma d\gamma = \int_{\gamma}^{\gamma_0} sec^3\gamma d(\tan\gamma)$ (Integrating by parts)

$= sec^3\gamma_0\tan\gamma_0 - \sec^3\gamma\tan\gamma - \int_{\gamma}^{\gamma_0} 3\sec^2\gamma\sec\gamma\tan\gamma\tan\gamma d\gamma = sec^3\gamma_0\tan\gamma_0 - sec^3\gamma\tan\gamma$

$- \int_{\gamma}^{\gamma_0} 3\sec^3\gamma(\sec^2\gamma - 1)d\gamma = sec^3\gamma_0\tan\gamma_0 - \sec^3\gamma\tan\gamma - 3\int_{\gamma}^{\gamma_0} sec^5\gamma d\gamma + 3\int_{\gamma}^{\gamma_0} sec^3\gamma d\gamma)$ see integral(7)

Or, $I_5 = -3I_5 + 3I_3 + sec^3\gamma_0 tan\gamma_0 - sec^3\gamma\, tan\gamma$

Or, $I_5 = \frac{3}{8}\{\sec\gamma_0\tan\gamma_0 + \log(\sec\gamma_0 + \tan\gamma_0) - \sec\gamma\tan\gamma - \log(\sec\gamma + \tan\gamma)\} + \frac{\sec^3\gamma_0\tan\gamma_0 - \sec^3\gamma\tan\gamma}{4}$ (16)

$h = \frac{1}{Cg}[\left\{\sec\gamma_0\tan\gamma_0 + \log(\sec\gamma_0 + \tan\gamma_0) + \frac{1}{CV_0^2\cos^2\gamma_0}\right\}\frac{(\tan^2\gamma_0 - \tan^2\gamma)}{2} - I_5 + 2I_3 - \frac{1}{2}[\{\{\log(\sec\gamma_0 + \tan\gamma_0)\}(\tan^2\gamma_0) - \{\log(\sec\gamma + \tan\gamma)\}(\tan^2\gamma) - \sec\gamma_0\tan\gamma + \sec\gamma\tan\gamma]$ (17)

Now we can proceed to find the path inclination at any instant of time and as such rewrite equation(3):

$$\frac{d\gamma}{dt} = \frac{-g\cos\gamma}{V} \quad (18)$$

Substituting the expression for V from (6) into the above equation and resorting to Binomial expansion and neglecting the square and other higher powers of the small terms as above, one gets

$$\frac{d\gamma}{dt} = -g\sqrt{c}[\left\{\sec\gamma_0\tan\gamma_0 + \frac{1}{V_0^2\cos^2\gamma_0} + \log(\sec\gamma_0 + \tan\gamma_0)\right\}^{\frac{1}{2}}\{1 - \frac{1}{2}\frac{\{\sec\gamma\tan\gamma + \log(\sec\gamma + \tan\gamma)\}]}{\sec\gamma_0\tan\gamma_0 + \frac{1}{V_0^2\cos^2\gamma_0} + \log(\sec\gamma_0 + \tan\gamma_0)}]\cos^2\gamma$$

$$\left\{\sec\gamma_0\tan\gamma_0 + \frac{1}{V_0^2\cos^2\gamma_0} + \log(\sec\gamma_0 + \tan\gamma_0)\right\}^{\frac{1}{2}} g\sqrt{c}dt = -\{1 + \frac{1}{2}.\frac{sec^2\gamma\{\sec\gamma\tan\gamma + \log(\sec\gamma + \tan\gamma)\}}{\sec\gamma_0\tan\gamma_0 + \frac{1}{V_0^2\cos^2\gamma_0} + \log(\sec\gamma_0 + \tan\gamma_0)}]d\gamma \qquad (19)$$

Integrating (16) and applying the initial conditions :at t=0, $\gamma = \gamma_0$,we get

$$\left\{\sec\gamma_0\tan\gamma_0 + \frac{1}{V_0^2\cos^2\gamma_0} + \log(\sec\gamma_0 + \tan\gamma_0)\right\}^{\frac{1}{2}} g\sqrt{c}t = \gamma_0 - \gamma + \frac{1}{2}.\frac{\frac{\sec^3\gamma_0 - \sec^3\gamma}{3} - \{(\tan\gamma)\log(\sec\gamma + \tan\gamma)\} + \{\tan\gamma_0(\log(\sec\gamma_0 + \tan\gamma_0)\} - \sec\gamma_0 + \sec\gamma}{\sec\gamma_0\tan\gamma_0 + \frac{1}{V_0^2\cos^2\gamma_0} + \log(\sec\gamma_0 + \tan\gamma_0)} \qquad (20)$$

which gives the time that elapses in researching the flight-path angle γ from its initial value γ_0.Hence parametric equations of the trajectory of the missile are given by (13) and (17). The greatest height attained,the corresponding horizontal distance covered and the time taken by the missile can be calculated by putting $\gamma = 0$ in the foregoing equations.However, the approximate equation of the trajectory ie a relation between horizontal distance X and altitude h can be determined by adopting a novel method of approximation:

A NOVEL METHOD OF APPROXIMATION TO DETERMINE RANGE AND ALTITUDE AND TRAJECTORY OF THE MISSILE

The range and altitude are functions of trigonometric circular functions $\tan\gamma$ and $\sec\gamma$ and logarithmic functions of $\tan\gamma$ and $\sec\gamma$.Since the difference $\epsilon(=\gamma_{0-}\gamma>0)$ is small as far as hitting the target or reaching a desired position by the missile,the foregoing functions can be expanded in Taylor's infinite series. Hence the range(13) and altitude (17) can respectively expressed as quadratic functions of ϵ neglecting its cubic and other higher powers in consequence of which,

$X= a_1 \epsilon^2+b_1 \epsilon +c_1,$ $\quad h= a_2 \epsilon^2+b_2 \epsilon +c_2,$ (21)

where a_1, b_1, c_1 and a_2, b_2, c_2 are constant and functions of V_0 , $\tan\gamma_0$ and $\sec\gamma_0$.

Elimination of ϵ^2 and ϵ between above two equations gives

$a_2X = a_1a_2 \epsilon^2+ b_1 a_2 \epsilon + c_1 a_2$, $\quad a_1h= a_1a_2 \epsilon^2+ b_2 a_1 \epsilon + c_2 a_1$

Or, $a_2X- a_1h=(b_1 a_2- b_2 a_1) \epsilon + c_1a_2 - c_2a_1$ (22)

$b_2X- b_1h=(b_2 a_1- b_1 a_2) \epsilon^2 + (c_1b_2 - c_2b_1)$ (23)

From (22),one gets

$$\epsilon^2 = \frac{b_2X-b_1h-(c_1b_2-c_2b_1)}{b_2a_1-b_1a_2} \quad (24)$$

$a_2X- a_1h=(b_1 a_2- b_2 a_1) \epsilon + c_1a_2 - c_2a_1$ (25)

Eliminating ϵ between (24) and (25) we get

$$\left(\frac{a_2X - a_1h - c_1a_2 + c_2a_1}{b_1a_2 - b_2a_1}\right)^2 = \frac{b_2X - b_1h - (c_1b_2 - c_2b_1)}{b_2a_1 - b_1a_2}$$

$$(a_2X - a_1h - c_1a_2 - c_2a_1)^2=(b_1a_2 - b_2a_1)(b_2X - b_1h - c_1b_2 + c_2b_1) \quad (26)$$

which can be simplified and rewritten as

$(a_2X - a_1h - A)^2$=B$(b_2X - b_1h - C)$ (27)

where A,B,C are constants and functions of a_i, b_i, c_i $(i = 1,2)$ and ultimately are functions of V_0, secγ_0 and tanγ_0.

Equation (27) represents a conic section and as such the trajectory of the missile for short interval of time is an ellipse, parabola or hyperbola depending upon the the conditions of its projection.The range R ie the horizontal distance described by the missile on reaching the horizontal plane is obtained by putting X=R and h=0 in equation (27):

$a_2^2R^2 - 2a_2RA + A^2$=B$(b_2R - C)$

Or, $a_2^2R^2 - (2a_2A + Bb_2)R + A^2 + BC$=0

Or,R = $\frac{(2a_2A+Bb_2)\pm\sqrt{(2a_2A+Bb_2)^2-4(A^2+BC)a_2^2}}{2a_2^2}$ (28)

which will give only one positive value as the range.

It can be visualized that there exists a maximum range for some value of γ_0.

REFERENCES

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