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On an Application of Laplace Transform to Solve a Simple Control System with Transfer Functions

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Abstract:

One of the principal tools applied in numerous areas such as Circuit Analysis, Control Systems, Communication System, Signal Processing, Mechanics, Hydrodynamics, Heat Conduction, Beam Problems, Networks and Space Theory is Laplace Transformation. Also in recent years, tremendous research efforts have been devoted to simple control system. In this article we transform jerk system as a simple example of the third-order control system with Laplace Transform and then we add the PID control to it and make it become a closed-loop system and get the transfer function.

Keywords **—**Laplace Transform,Inverse Laplace Transform,Jerk system, PID control

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I. INTRODUCTION

Transformations in mathematics are applied for a similar reason that transformations in real life are,and that is to make something easier to handle.If you had to carry a large amount of natural gas, it'd be much easier to compress/liquefy it and carry that instead. And when you reach the destination, you can recover the gas again. The process of applying Laplace transforms is pretty much the same. You apply the transform to a differential equation and then turn it into an algebraic equation, thus making it significantly easier to *handle*. Then you carry out the simplification of the algebraic expression to the required extent. Now you find the inverse Laplace transform of simpler expression(s) which solves the differential equation.Although this is one way of solving differential equations through an algebraically manageable equation, the "daily life" applications for Laplace transformations is concentrated engineering.

Among the various integral transform mostly used,Laplace Transform is first introduced byPierre-Simon Laplace, a French Mathematician. He introduced a special type of integral transform in his research, later on it was called as Laplace transformation. Then Oliver Heaviside, a British physicist, developed Laplace transformation systematically. The easiness in understanding and simple in applying is the inspiration behind this transformation technique. In numerous problems Laplace transformation is applied to derive the general solution.

Laplace transformations can be used whether around you're an engineer analysing circuits, analysing systems of heating and air conditioning, or construction and building. The Laplace transform is particularly useful in solving linear ordinary differential equations such as those arising in the analysis of electronic circuits; control system etc. [2].A control system manages commands, directs or regulates the behaviour of other devices or systems. It can range from a home heating controller using a thermostat controlling a domestic boiler to large Industrial control systems which are used for controlling processes or machines. The Laplace transform converts the governing differential equations of a system or its components into simple algebraic form allowing the controls engineer to describe the system, in particular a closed loop system, as a chain of connected functional blocks also called a block diagram.[5],[7]. Laplace Transformations helps to find out the current and some criteria for the analysing the circuits. It is used to build required ICs and chips for systems. So it plays a vital role in field of computer science.

II. Basic Concept

Jerk:It is known to all that the first derivative of **position** (symbol *x*) with respect to time is **velocity** (symbol ***v***), and the second derivative is **acceleration** (symbol ***a***). Less well known is that the third derivative, i.e. the rate of increase of acceleration, is technically known as *jerk* ***j***. Jerk is a vector, but may also be used loosely as a scalar quantity because there is not a separate term for the magnitude of jerk analogous to **speed** for magnitude of velocity.In the UK, *jolt* has sometimes been used instead of jerk, and is equally acceptable. Many other terms have appeared in individual cases for the third derivative, including pulse, impulse, bounce, surge, shock and super acceleration. These are generally less appropriate than jerk and jolt.It is also recognized in international standards: In ISO 2041 (1990), Vibration and shock - Vocabulary, page 2:"1.5 jerk: A vector that specifies the time-derivative of acceleration."
Note that the symbol ***j*** for jerk is not in the standard and is probably only one of many symbols used.[8]

As its name suggests, jerk is important when evaluating the destructive effect of motion on a mechanism, or the discomfort caused to passengers in a vehicle. The movement of delicate instruments needs to be kept within specified limits of both acceleration and jerk to avoid damage. When designing a train, the engineers will typically be required to keep the jerk less than 2 meters per second cubed for passenger comfort. In the aerospace industry they even have such a thing as a *jerk meter*: an instrument for measuring jerk.

Laplace transform: plays a important role in control theory as most control system analysis and design techniques are based on linear systems theory.It appears in the description of linear time invariant systems, where it changes convolution operators into multiplication operators and allows to define the transfer function of a system. The properties of systems can be then translated into properties of the transfer function.To create transfer functions, we need the notion of the Laplace transform. Basically The Laplace transform of a time-domain function, *f*(*t*), is represented by *L*[*f*(*t*)] and is defined as $L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,dt$.[1]Where the parameter s may be real or complex,The Laplace transform of $f(t)$ is said to be exist if the integral converge for some value of s. Otherwise is does not exist.

III. HOW TO DO LAPLACE TRANSFORM

In brief, if we have $f(t)$, which is the function of time, we can use Laplace integral:

$$F(s) = \int_0^\infty f(t)\, e^{-st} dt$$

To transfer the time domain t to the frequency domain s; s is a complex number. It should be clear that what we use is the **one-sided Laplace transform** which corresponds to $t \geq 0$ (all non-negative time). Now focus on how to do Laplace transforms.
Given some examples, if $f(t) = 1$, which is a constant function of time. Then what is its Laplace transform?

$$L\{1\} = \int_0^\infty e^{-st}dt$$

$$= [-\frac{1}{s}e^{-st}]_0^\infty$$

$$= 0 - (-\frac{1}{s})$$

$$= \frac{1}{s}$$

Let's see more examples to get some ideas about Laplace transform if $f(t) = t$, what is its Laplace transform?

Before calculating this integral, according to Integration Rules:

$$\int uv' = uv - \int u'v$$

We know $f(t) = t$,so we can let ,

$$u = t, v = -\frac{1}{s}e^{-st}$$

And we know,

$$u' = 1, v' = e^{-st}$$

According to the definition of Laplace transform and Integration rules, we can arrive:

$$L\{t\} = \int_0^\infty t^* e^{-st} dt$$

$$= [t^* (-\frac{1}{s}e^{-st})]_0^\infty \quad - \int_0^\infty -\frac{1}{s}e^{st} dt$$

$$= 0 + \frac{1}{s}L\{1\} = \frac{1}{s^2}$$

Now, let's see some more general properties of the Laplace transform. For example, what is the Laplace transform of $f'(t)$ which is the derivative of $f(t)$?

$$L\{f'(t)\} = \int_0^\infty e^{-st} * f'(t)dt$$

As we did before, We can also use Integration rules , this time we let:

$$u = e^{-st}, v = f(t)$$

We can get $L\{f'(t)\}$ as follows:

$$L\{f'(t)\} = \int_0^\infty e^{-st} * f'(t)dt \quad = \int_0^\infty uv'\, dt \quad = uv - \int_0^\infty u'vdt$$

$$= [e^{-st} f(t)]_0^\infty - \int_0^\infty (-s)e^{-st} f(t)dt$$

$$= 0 - f(0) + s\int_0^\infty e^{-st} f(t)dt$$

$$= sL\{f(t)\} - f(0)$$

Furthermore,We can get the Laplace transform of the second derivative of $f(t)$, Which is $f''(t)$: and third derivative of $f(t)$, Which is $f'''(t)$

$$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$$

And $L\{f'''(t)\} = s^3 L\{f(t)\} - s^2 f(0) - sf'(0) - f''(0)$

Processing in this way for nth derivatives we get $L[f^n(t)] = s^n L[f(s)] - s^{n-1} f(0) - sf'(0) - ...... - f^{n-1}(0)$

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2}{s^3}$
$t^n, (n = 1, 2, 3 ...)$	$\frac{n!}{s^{n+1}}$
t^a, (a is positive)	$\frac{\Gamma(a+1)}{s^{a+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$

Fig.1 some useful formula for Laplace Transform

IV. Solving The Jerk System with Laplace Transform

We know that Laplace transform is used to transfer differential equations to algebraic equations, which can then be solved by the formal rules of the algebra. Let's see a simple example first.

Let the dynamic equation of the Jerk system is

$$x''' + Ax'' + x' - |x| = F$$

Given initial state $x(0) = 0,\ x'(0) = 0$ and Here, A is an adjustable parameter.

But now, we cannot solve this equation as it is a differential equation in the time domain. We can solve this differential equation like solving an algebraic problem with Laplace transform as it transform this from the time domain to the frequency domain and arrive at an algebraic equation. This sounds hard to imagine. Let's see how transforms is!

Back to our example above, we want to know the open-loop step response of the Jerk System,for example we use the following jerk system

$$\frac{d^3x}{dt^3} + A\frac{d^2x}{dt^2} + \frac{dx}{dt} - |x| = F$$

we can write as $x''' + Ax'' + x' - |x| = F$

Which we cannot solve easily but if we transfer it to s domain, we can get the result.

Fig.2 Open-loop step response system

According to the results of part2, we know the first, second and third derivatives of $F(t)$ is as follows:

$$x''' + Ax'' + x' - x = F$$

So we can arrive at $x' = s * X(s) - X(0)$ we denoted $X(s)$ as the Laplace transform of function $X(t)$. Given $X(0) = 0$ and $X'(0) = 0$ so,

$$L\{x'\} = s * X(s)$$

$$L\{x''\} = s^2 * X(s) - s * x(0) - x'(0)$$

$$L\{x'''\} = s^3 * X(s) - s^2 * x(0) - s * x'(0) - x''(0)$$

Now applying the condition, we transfer the dynamic model as follows

$$s^3 * X(s) + A\{s^2 * X(s)\} + s * X(s) - X(s) = F(s)$$

Finally, we get the transfer function G(s) of this control system. The definition of the transfer function of a control system is its outputs divided its inputs. In this case, X(s) is the output; $F(s)$ is the input, so we can get $G(s)$ as follows:

$$or, X(s)(s^3 + As^2 + s - 1) = F(s)$$

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{(s^3 + As^2 + s - 1)}$$

Suppose the input F =1, A=-1, we can get the output $X(s)$ as follows:

$$\therefore X(s) = \frac{1}{(s^3 - s^2 + s - 1)} = \frac{1}{(s-1)(s^2+1)}$$

$$\therefore X(s) = \frac{1}{2(s-1)} - \frac{s}{2(s^2+1)} - \frac{1}{2(s^2+1)}$$

The last step is taking the **inverse transform** then gives,

$$x(t) = \frac{1}{2}(e^t - \cos t - \sin t)$$

If we want to control this system, one method is to use, the PID control which can convert this problem to be a closed-loop system.[4][7]

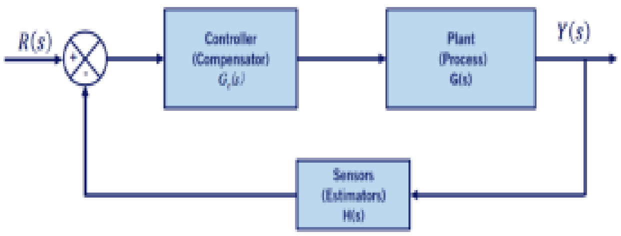

Fig.3 Closed loop system

The notation is this: G(s) is the plant transfer function, which we have solved before. $G_c(s)$ isthe controller transfer function, $R(s)$ is a reference (or command) input, and Y(s) is the output.

For the unity feedback, i.e., H(s) =1, the closed-loop system is given by,

$$\frac{Y(s)}{R(s)} = \frac{G(s)G_c(s)}{1+G(s)G_c(s)}$$

According to the above example, we know the plant transfer function is, $G(s) = \frac{1}{(s^3 + As^2 + s - 1)}$

Meanwhile, we know that the block diagram of PID control as below,[4],[7]

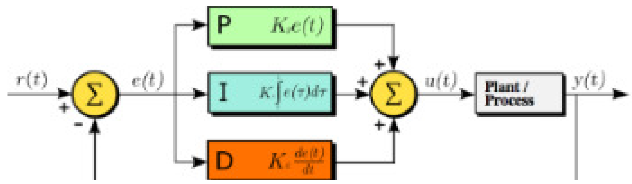

Fig.4 PID Control

According know its transfer function in the time domain is,

$$u(t) = K_p e(t) + K_i \int_0^t e(t')dt' + K_p \frac{de(t)}{dt}$$

Where Kp, Ki, Kd all non-negative, denote the coefficients for the proportional, integral and derivative terms respectively. $e(t)$ is the error.[4][7]

In the control system, we must transfer all functions into the Laplace domain. According to part2, we know how to transfer a function from the time domain (t) to the frequency domain(s). In this case, let $L\{e(t)\} = 1$,
According to part2, we know that

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

Suppose $e(0) = 0, so\ L\{e'(t)\} = s*1 - e(0) = s$, equivalently,

$$L\{\int_0^t e(t')\,dt'\} = \frac{1}{s}$$

The transfer function in the Laplace Domain of the PID controller is,

$$L(s) = K_p + Ki/s + K_d s$$

Our controller transfer function can be written as,

$$G_c(s) = K_p + \frac{K_i}{s} + Kd * s$$

Above all, the transfer function of the closed-loop system with PID control gives,

$$\frac{Y(s)}{R(s)} = \frac{G(s)G_c(s)}{1+G(s)G_c(s)}$$

$$= \frac{\{sK_p + K_i + K_d s^2\}}{s^4 - s^3 + s^2(1+k_d) + s(k_p - 1) + k_i}$$

V.Conclusion:

This article illustrates a simple example of the Third-order control system and goes through how to solve it with Laplace transform. Furthermore, we add the PID control to it and make it become a closed-loop system and get the transfer function step by step. The Laplace transform plays a important role in control theory as most control system analysis and design techniques are based on linear systems theory.So we can also solve third order control system and so on in many of our real life problem like AC induction motor controlling, DC motor speed control in a cooling fan etc.

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