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Convergence of Some Special Numerical Series

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Abstract:

The summation of infinite series is a difficult problem that has puzzled mathematicians for several centuries. In this paper, several series based on $\{\frac{1}{n}\}$ are selected to explore different convergence and divergence proof of series.

Keywords **—Convergence, Divergence, Series**

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I. DEFINITION OF SERIES CONVERGENCE

Firstwe give a sequence $\{a_n\}$, we use the notation $\sum_{n=p}^{q} a_n (p \le q)$ to denote the sum $a_p + a_{p+1} + \cdots + a_q$.With $\{a_n\}$ we associate a sequence $\{s_n\}$, where $s_n = \sum_{k=1}^{n} a_k$. If $\{s_n\}$ converges to$s$, we say that the series converges, and write $\sum_{k=1}^{\infty} a_k = s$.The number $s$is called the sum of the series, and it should be clearly understood that $s$is the limit of a sequence of sums. If $\{s_n\}$ diverges, the series is said to diverge.

II. CONVERGENCE OF SOME SERIES BASED ON $\{\frac{1}{n}\}$

Example 1. Consider the convergence of the following series

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots$$

Proof.

Let $p = m$, we see that

$$\begin{aligned}|u_{m+1}+u_{m+2}+\cdots+u_{2m}| &= \left|\frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{2m}\right| \\ &\ge \left|\frac{1}{2m}+\frac{1}{2m}+\cdots+\frac{1}{2m}\right| = \frac{1}{2}.\end{aligned}$$

Given $\varepsilon_0 = \frac{1}{2}$, for any $N > 0$, there exists $m > N$ and $p = m$, such that $\sum_{n=m}^{2m} u_n > \varepsilon_0$. By the Cauchy criterion, we can have the series is diverge.

Example 2. Consider the convergence of the following series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+(-1)^{n-1}\frac{1}{n}+\cdots$$

Proof.

Given a power series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$, it converges for all other x with $-1 < x \le 1$. And $S(x)$ is the sum of the power series, we have $S(x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$.

Since

$$\begin{aligned}S(x) &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} \\ &= \sum_{n=1}^{\infty}(-1)^{n-1}\int_0^x \left(\frac{t^n}{n}\right)_t' dt \\ &= \int_0^x \left[\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{t^n}{n}\right)_t'\right] dt \\ &= \int_0^x \left[\sum_{n=1}^{\infty}(-1)^{n-1} t^{n-1}\right] dt = \int_0^x \frac{1}{1+t} dt \\ &= \ln(1+x), \qquad x \in (-1,1]\end{aligned}$$

We have $S(1) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = ln2$.

For simplicity, we note $\sigma_n = \sum_{k=1}^{n}\frac{1}{n}$and $l_n =$

$\sum_{k=1}^{n}(-1)^{k-1}\frac{x^k}{k}$. Another more accurate description of $\sum_{k=1}^{n}\frac{1}{n}$ is $\sum_{k=1}^{n}\frac{1}{n}=\ln n+c+r_n, r_n\to 0(n\to\infty), and c is a const.$

Example 3. Consider the convergence of the following series

$$1-\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots$$

Proof.

Since $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+(-1)^{n-1}\frac{1}{n}+\cdots$, then $\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots+(-1)^{n-1}\frac{1}{2n}+\cdots$, hence we add the above two equations, we have $1-\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots=\frac{3}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}=\frac{3}{2}ln2.$

Example 4. Consider the convergence of the following series

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\cdots$$

Proof.

Since $S_{3n}=1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\cdots+\frac{1}{2n-1}-\frac{1}{4n-2}-\frac{1}{4n}=\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right)-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{4n}\right)=\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n-1}+\frac{1}{2n}\right)-\left(\frac{1}{2}+\cdots+\frac{1}{2n}\right)-\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{4n}\right)=\sigma_{2n}-\frac{1}{2}\sigma_n-\frac{1}{2}\sigma_{2n}=\frac{1}{2}(\sigma_{2n}-\sigma_n)=\frac{1}{2}l_n$, and $\lim_{n\to\infty}l_n=ln2$, we have $\lim_{n\to\infty}S_{3n}=\frac{1}{2}ln2$ .Hence $\lim_{n\to\infty}S_{3n+1}=\frac{1}{2}ln2$, $\lim_{n\to\infty}S_{3n+2}=\frac{1}{2}ln2$. On the whole, $\lim_{n\to\infty}S_n=\frac{1}{2}ln2$.

Example 5. Consider the convergence of the following series

$$1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}+\frac{1}{8}-\frac{1}{9}+\cdots$$

Proof.

We see that $|x_{3n+1}+x_{3n+2}+\cdots+x_{6n}|=\frac{1}{3n+1}+\frac{1}{3n+2}-\frac{1}{3n+3}+\frac{1}{3n+4}+\frac{1}{3n+5}-\frac{1}{3n+6}+\cdots+\frac{1}{3n+(3n-2)}+\frac{1}{3n+(3n-1)}-\frac{1}{3n+3n}>\frac{1}{3n+1}+\frac{1}{3n+4}+\cdots+\frac{1}{6n-2}>\frac{3n}{6n-2}=\frac{1}{6}$.Given $\varepsilon_0=\frac{1}{6}$, for any $N>0$, there exists $m>N$ and $p=3m$, such that $|\sum_{n=3m+1}^{6m}x_n>|\varepsilon_0$.By the Cauchy criterion, we can have the series is diverge.

Example6. Consider the convergence of the following series

$$1+\frac{1}{2}+\frac{1}{3}-\frac{1}{4}-\frac{1}{5}-\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}-\cdots$$

Proof.

If $S_1'=1-\frac{1}{4}+\frac{1}{7}-\frac{1}{10}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^n}{3n+1}, S_2'=\frac{1}{2}-\frac{1}{5}+\frac{1}{8}-\frac{1}{11}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^n}{3n+2}, S_3'=\frac{1}{3}-\frac{1}{6}+\frac{1}{9}-\frac{1}{12}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{3n}$,by the Leibniz theorem, we can have these three series are converge, so the default series is converge.

Example 7. Consider the convergence of the following series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\cdots$$

Proof.

Given a power series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$, it converges for all other x with $-1\le x\le 1$. Since $\frac{1}{1+x^2}=1-x^2+x^4-x^6+\cdots, x\in(-1,1)$, we see that $\arctan x=\int(1-x^2+x^4-x^6+\cdots)dx=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$, by the Abel theorem, we can have that

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\cdots=\lim_{x\to 1-0}arctanx=arctan1=\frac{\pi}{4}.$$

Example 8. Consider the convergence of the following series

$$1+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}\cdots$$

Proof.

For Fourier $f(x)=\begin{cases}-\frac{\pi}{4} & -\pi<x<0\\ \frac{\pi}{4} & 0\le x<\pi\end{cases}$, $f(x)=\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}$. When $x=\frac{\pi}{4}$, we have $f\left(\frac{\pi}{2}\right)=\frac{\pi}{4}$, so $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\cdots$.

It follows that $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\left(1+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\cdots\right)+\left(-\frac{1}{3}+\frac{1}{9}-\frac{1}{15}+\cdots\right)=\left(1+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\cdots\right)+\left(-\frac{1}{3}\right)\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\cdots\right)=\left(1+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\cdots\right)-\frac{1}{3}\frac{\pi}{4}$, so $1+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\cdots=\frac{\pi}{3}$.

Examples 9. Consider the convergence of the following series

$$\sum_{n=1}^{\infty}\frac{(-1)^{[\sqrt{n}]}}{n}$$

Proof.

$$\sum_{n=1}^{\infty}\frac{(-1)^{[\sqrt{n}]}}{n}=-1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}-\frac{1}{9}-\cdots-\frac{1}{15}+\cdots$$

The former series can be rewritten as $\sum_{k=1}^{\infty}(-1)^k a_k$, where $a_k=\frac{1}{k^2}+\frac{1}{k^2+1}+\cdots+\frac{1}{k^2+2k}$. On the one hand, $a_k=\frac{1}{k^2}+\frac{1}{k^2+1}+\cdots+\frac{1}{k^2+2k}>\frac{1}{k^2+k}+\frac{k+1}{k^2+2k+1}=\frac{2}{k+1}\to 0(k\to\infty)$, on the other hand, $a_k=\frac{1}{k^2}+\frac{1}{k^2+1}+\cdots+\frac{1}{k^2+2k}<\frac{k}{k^2}+\frac{k+1}{k^2+k}=\frac{2}{k}\to 0(k\to\infty)$, by the squeeze theorem, we have $\lim_{k\to\infty}a_k=0$,and $a_{k+1}<\frac{2}{k+1}<a_k$, the series $\{a_k\}$ is monotonic decreasing, by the Leibniz theorem, we can get $\sum_{k=1}^{\infty}(-1)^k a_k$ is converge, thus $\sum_{n=1}^{\infty}\frac{(-1)^{[\sqrt{n}]}}{n}$is converge.

Since $\left|\sum_{n=1}^{\infty}\frac{(-1)^{[\sqrt{n}]}}{n}\right|=\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so $\sum_{n=1}^{\infty}\frac{(-1)^{[\sqrt{n}]}}{n}$converges non-adsolutely.

Examples 10. Rearrange the series of $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$, first there are p positive terms, then there are q negative terms, and so on, then the new series converges, and the sum is $\ln 2+\frac{1}{2}\ln\frac{p}{q}$.

Proof .

The Rearranged series can be written as

$$\left(1+\frac{1}{3}+\cdots+\frac{1}{2p-1}\right)-\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2q}\right)+\left(\frac{1}{2p+1}+\cdots+\frac{1}{4p-1}\right)-\left(\frac{1}{2q+2}+\cdots+\frac{1}{4q}\right)+\cdots$$

Now we denote $1+\frac{1}{3}+\cdots+\frac{1}{2m-1}=H_{2m}-\frac{1}{2}H_m$, $\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2m}=\frac{1}{2}H_m$.

$$S_{2n}=\left(1+\frac{1}{3}+\cdots+\frac{1}{2p-1}\right)-\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2q}\right)+\cdots+\left(\frac{1}{2(n-1)p+1}+\cdots+\frac{1}{2np-1}\right)-\left(\frac{1}{2(n-1)q+2}+\cdots+\frac{1}{2nq}\right)=\left(H_{2np}-\frac{1}{2}H_{np}\right)-\frac{1}{2}H_{nq}$$

When $n\to\infty$,$S_{2n}\to\ln 2+\frac{1}{2}\ln\frac{p}{q}$.

III. CONCLUSIONS

Series and its summation is a new form of studying series and its limit. This new form enriches and develops the content and method of studying series，It provides a powerful tool for the further study of function series and other mathematical theories. This paper give the proofs of some series based on$\{\frac{1}{n}\}$, varied proofs reveal the diversity of the problem.

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