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A Study on Linear Algebra, Vector Sapces and Subspaes

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Abstract:
The paper is a study of linear on the linear algebra, vector spaces and subspaces .Linearalgebradeals withthestudyofvectors, vector spaces, sub spaces, linear combinations, lines and spaces , linear transformations, system of equations. Linear algebra is prerequisite to a deeper understanding of machine learning. Linear algebra is also used in most sciences and fields of Engineering. It is one of the most central topics of mathematics. Most modern geometrical concepts are based on linear algebra.

Keywords—**Vectors,Vectorspace,subspace,setandsubset.**

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I.INTRODUCTION:

The general linear equation isrepresented as$a_1x_1 + a_2x_2\ldots +a_nx_n = b$, Here,a's – represents the coefficients, x's – represents the unknowns andb – represents the constant.If there exists a system of linear algebraic equations, which is the set of equations, then can be solved using the matrices.

Definition1.Let F be a field and V be a non-empty set defined under two binary operations addition and scalar multiplication, then V is said to be a vector space if the following axioms are satisfied.

V is an abelian group under addition, i.e.,

α, β∈V, α+β∈V (closure axiom)

(α+β)+γ= α+(β+γ) ∀ α,β,γ∈V (Associative law)

∃0∈V,∋α+0=0+α=α.(Identity axiom)

∀α∈V, ∃-α∈V such that α+(-α)=(-α)+α=0(Inverse axiom)

α+β=β+α, ∀α,β∈V(commutative axiom)
a∈F, α∈V implies aα∈V, ∀a∈F

a(α+β)=aα+aβ, ∀a∈F and α,β∈V

(a+b)α=aα+bα, ∀a,b∈F and α∈V

(ab) α=a(bα)∀a,b∈F and α∈V

∃1∈F such that 1.α=α.1=α, ∀α∈V[1, p.207].

The elements of V are vectors and elements of F are scalars. The identity element of the group V under addition is denoted by '0' and is called zero vector or null vector, which is unique. The null vector zero should not be confused with the scalar '0'. The space {0} is called zero space or null space [4, p.37].

Example: Let V be the set of all 2x2 matrices over real numbers. We claim that V is a vector space.

First, we prove that V is an abelian group under addition.

Since addition of any two 2x2 matrices over real numbers is again a 2x2 matrix over real numbers, V is closed under addition.

Since matrix addition is always associative, associative axiom is satisfied.

Zero matrix of order 2x2 in V, is the additive identity, identity axiom is satisfied
∀ α∈V, ∃-α∈V ∋ α+(-α)=(-α)+α=0, inverse axiom

is satisfied.

Since matrix addition is always commutative, commutative axiom is satisfied.

V is an abelian group under addition.

Further, If $a\in F$, and $\alpha\in V$, then $a\alpha$ is also a matrix of order 2x2 with elements as real numbers.

V is closed under scalar multiplication.

$a(\alpha+\beta)=a\alpha+a\beta$, $\forall a\in F$ and $\alpha,\beta\in V$

$(a+b)\alpha=a\alpha+b\alpha$, $\forall a,b\in F$ and $\alpha\in V$

$(ab)\ \alpha=a(b\alpha)\forall a,b\in F$ and $\alpha\in V$

$\exists 1\in F$ such that $1.\alpha=\alpha.1=\alpha$, $\forall\alpha\in V$

V is a vector space over real numbers.

Definition2. A non-empty sub set w of a vector space V over a field F is called a subspace of V if w itself is a vector space over F under the same operations of addition and scalar multiplication as defined in V.

Zero space and vector space V are subspaces of V over F and are called improper subspaces or trivial subspaces of V, all other subspaces of V are called proper subspaces of V.

A non-empty sub set w of a vector space V is a subspace of V iffi) w is closed under addition and ii) w is closed under scalar multiplication.

Proof: Suppose w is, a subspace of V, then w is a vector space over F, under the operations ofaddition and scalar multiplication as defined in V. Hence, w isclosed under and scalar multiplication.

Conversely,
Suppose w satisfies (i) and (ii), we shall show that w is a subspace of v

By condition (i), $\forall\alpha,\beta\in w$, then $\alpha+\beta\in w$(closure axiom)

By condition (ii) $\forall c\in F$ and $\alpha\in w, c\alpha\in w$
In particular if $C=0\in F$, $\alpha\in w$, $0.\alpha=0\in w$ (identity axiom)
if $C=-1\in F$, $\alpha\in w$, $(-1)\ \alpha=-\alpha\ \in w$ (inverse axiom).

Further as addition is commutative and associative in V so in w

(w,+) is an abelian group.

The other axioms of the vector space hold in w as they hold in V
Hence w is a vector space over F and therefore a subspace of V.To verify W is a subspace of V, we are required to check that W is closed under vector addition and scalar multiplication.[3,p.13]

Further, A non-empty sub set w is a sub space of a vector space V over F iff $a\alpha+b\beta\in w$ $\forall\alpha,\beta\in w$ and $a,b\in F$
Proof: Let w be a sub space of V.
Let $a, b\in F$ and $\alpha, \beta\in w$
$a\alpha, b\beta\in w$ and hence $a\alpha+b\beta\in w$.

Conversely,
Let $a\alpha+b\beta\in w\ \forall\ \alpha,\beta\in w$ and $a,b\in F$
Let a=1, b=1, then $1.\alpha+1.\beta=\alpha+\beta\in w\ \forall\ \alpha,\beta\in w$
w is closed under vector addition.

Now, take b=0, $a\alpha+o\beta=a\alpha\in w$. $\forall a\in F$ and $\alpha\in w$
w is closed under scalar multiplication.

w is a sub space of V.

The intersection of any two-sub spaces of a vector space V over a field F is also a sub space of V.
Proof: Let S and T be two sub spaces of a vector space V over the field F
Since $0\in S$ and $0\in T$ hence $0\in S\cap T$ $S\cap T\neq\emptyset$
Let $\alpha,\beta\in S\cap T$.
Let $\alpha,\beta\in S$, $\alpha,\beta\in T$.

$a\alpha+b\beta\in S$ and $a\alpha+b\beta\in T$ $\forall a,b\in F$.

$a\alpha+b\beta\in S\cap T$.

$S\cap T$ is a sub space of V [2, p.170].

Further, the union of two sub spaces need not be a sub space of V.

Proof: Let R be the field of real numbers

$W_1=\{(0,0,z)/z\in R)$ and $W_2=\{(0,y,0)/y\in R\}$ are two sub spaces of V_3 (R).

Let $(0,0,3)\in W_1$, $(0,5,0)\in W_2$

$(0,0,3)+(0,5,0)=(0,5,3)$ which is neither belong to W_1 nor to W_2.

$W_1 \cup W_2$ is not closed under addition

$W_1 \cup W_2$ is not a sub space of V_3 (R).

If W_1 and W_2 are sub spaces of a vector space V(F), then $W_1 + W_2$ is also a vector space of V(F).

Proof: Let $\alpha, \beta \in W_1 + W_2$ be any two arbitrary elements.

Then $\alpha=\alpha_1+\alpha_2$, $\beta=\beta_1+\beta_2$ where $\alpha_1,\beta_1 \in W_1$, and $\alpha_2,\beta_2 \in W_2$.

Let $a,b \in F$, $a\alpha+b\beta=a(\alpha_1+\alpha_2)+b(\beta_1+\beta_2)$

$= (a\alpha_1+b\beta_1)+ (a\alpha_2+b\beta_2)$….(1)

Since W_1 and W_2 are sub spaces of V, $a,b \in F$, $\alpha_1,\beta_1 \in W_1 \Rightarrow a\,\alpha_1+\beta_1 \in W_1$

$a, b \in F$, $\alpha_2,\beta_2 \in W_2 \Rightarrow a\alpha_2+b\beta_2 \in W_2$

Hence, $a\alpha+b\beta \in W_1+ W_2$

$a,b \in F$ and $\alpha, \beta \in W_1+ W_2$, implies $a\alpha+b\beta \in W_1+ W_2$

W_1+W_2 is a sub space of V(F).

Conclusion:

To be a subspace of a given vector space a non-empty-subset of the vector space is to be closed under the same operations of addition and scalar multiplications as defined in the vector space. Further, the intersection of two subspaces of a vector space is again a subspace of the vector space and the union of the two subspaces of a vector space need not be a subspace of the vector space, the sum of two subspaces of a vector space is again a sub space of the vector space.

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