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A New Numerical Solutions of Fractional Differential Equations with Atangana-Baleanu operator in Reimann sense

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Abstract:

Fractional calculus techniques are effectively used in many fields of science and engineering, one of the techniques is the Sumuduhomotopy permutation method (SHPM), which researchers did not study with the fractional derivative of Atangana-Baleanu in Reimann sense (ABRO). This work aims to study the Sumuduhomotopy permutation method (SHPM) to solve fractional differential equations (FDEs) with ABRO. We introduce the algorithm of this technique with ABRO, the method has been applied to FDEs with the ABRO. To conclude, the method effectively solved this type of fractional differential equations.

Keywords **—Sumudu Transform ; Atangana-Baleanu Operator; Homotopy Permutation Method; Fractional Differential Equation.**

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I. INTRODUCTION

Approaches for fractional calculus are widely utilized in engineering and science.Researchers have recently been increasingly interested in fractional derivatives, particularly because various applications in medicine, economics, physics, and technology have emerged.There were both local and nonlocal definitions of fractional derivatives offered. Nonlocal derivatives are more intriguing since the majority of these applications are dependent on the function's background. Several derivatives are offered based on singular kernels, like the Caputo, Riemann-Liouville, and Grünwald formulations.Recently, several fractional derivative definitions built on nonsingular kernels, for instance, the Caputo-Fabrizio and Atangana-Baleanu fractional derivatives, were presented [1,2].

Nonlinear differential equations may be used to explain the majority of natural occurrences. Therefore, experts working in many fields of research endeavor to find solutions. It is challenging to obtain a precise solution to these sets of equations due to their nonlinear nature, Therefore, researchers use approximate methods for example, FADM, FLADM, FSADM, FNADM, FEADM, FHPM, FLHPM, FSHPM, FNHPM, FEHPM, FVIM, FLVIM, FSVIM, FNVIM and FEVIM to solve differential equations [3-13].

In this study, the fractional partial differential equations containing the fractional operator of the fractional operator ABC are solved using SHPM. The order of the paper is as follows: Section 2 presents the fundamental concepts of calculus and fractional integration. Section 3 analyzes the techniques utilized. Section 4 provides several instances that demonstrate the efficacy of the approach described. Section 5 provides the conclusion 5.

II. BASIC CONCEPTS

Def. 1 [14] Let $\hbar \in H^1(\sigma_1, \sigma_2), \sigma_1 > \sigma_2$, the Atangana-Baleanu operator in Reimann sense for

$0 < \mathbb{Q} < 1$ is,

$$ {}^{ABR}\mathcal{D}_{\xi}^{\mathbb{Q}} \hbar(\xi) = \frac{\mathcal{B}(\mathbb{Q})}{1-\mathbb{Q}} \frac{d}{d\xi} \int_0^{\xi} \mathrm{E}_{\mathbb{Q}} \left(-\frac{\mathbb{Q}(\xi-\mathcal{E})^{\mathbb{Q}}}{1-\mathbb{Q}} \right) \hbar(\mathcal{E}) d\mathcal{E}, \xi \geq 0, \quad (1) $$

where $\mathcal{B}(\mathbb{Q})$is a normalization function such that $\mathcal{B}(0) = \mathcal{B}(1) = 1$.

Def. 2 [15] The Sumudutransform is defined over the set of function

$$ \mathcal{A} = \left\{ \hbar(\xi) \middle| \begin{array}{c} \exists \mathcal{M}, \tau_1, \tau_2 > 0, |\hbar(\xi)| < M e^{\frac{|\tau|}{\tau_j}}, \\ if\ \tau \in (-1)^j \times [0,\infty) \end{array} \right\}, $$

by the following formula

$$ \mathcal{S}[\hbar(\xi)] = \mathcal{G}(s) = \int_0^{\infty} \hbar(s\xi) e^{-\xi} d\xi, \ s \in (\tau_1, \tau_2). \quad (2) $$

Def. 3 [16] The inverse Sumudu transform of a function is defined by

$$ \mathcal{S}^{-1}[\mathcal{G}(s)] = \hbar(\xi) = \frac{1}{2i\pi} \int_{p-\infty}^{p+\infty} e^{\frac{\xi}{s}} \mathcal{G}(s)\, d\xi, \ s > 0, \quad (3) $$

where s is Sumudu transformation variable and p is a real constant.
The Sumudu transformation of Atangana-Baleanu-Reimann derivative can be obtained by [17],

$$ \mathcal{S}({}^{ABR}_{\ \ 0}\mathcal{D}_{\xi}^{\mathbb{Q}} \hbar(\xi)) = \frac{\mathcal{B}(\mathbb{Q})}{1-\mathbb{Q}+\mathbb{Q}s^{\mathbb{Q}}} \mathcal{G}(s). \quad (4) $$

Some properties of Sumudu transform [18],

$$ \mathcal{S}[\beta] = \beta. $$
$$ \mathcal{S}[\xi\xi] = s. $$
$$ \mathcal{S}[\xi^n] = n!\, s^n. $$
$$ \mathcal{S}[e^{\beta\xi}] = \frac{1}{1-\beta s}. $$
$$ \mathcal{S}[\sin(\beta\xi)] = \frac{\beta s}{1+\beta^2 s^2}. $$
$$ \mathcal{S}[\cos(\beta\xi)] = \frac{s}{1+\beta^2 s^2}. $$

III. Analysis of the Method

Considering the following FPDEs with ABRO

$$ {}^{ABR}\mathcal{D}_{\xi}^{\mathbb{Q}} \hbar(z,\xi) + \mathrm{O}_1[\hbar(z,\xi)] + \mathrm{O}_2[\hbar(z,\xi)] = g(z,\xi), \quad (5) $$

with initial condition $\hbar(z,0) = \hbar_0(z)$,
where${}^{ABR}\mathcal{D}_{\xi}^{\mathbb{Q}}$is ABRO , $\mathrm{O}_1$is a linear operator, $O_2$is a nonlinear operator and $g$is a source term.

Applying ST to both side of Eq.(5) with the initial condition,

$$ \frac{\mathcal{B}(\mathbb{Q})}{1-\mathbb{Q}+\mathbb{Q}s^{\mathbb{Q}}} \mathcal{G}(s) = \mathcal{S}(g(z,\xi) - \mathrm{O}_1[\hbar(z,\xi)] - \mathrm{O}_2[\hbar(z,\xi)]), \quad (6) $$

substituting initial condition of Eq.(5), Eq.(6) becomes

$$ \mathcal{G}(s) = -\frac{1-\mathbb{Q}+\mathbb{Q}s^{\mathbb{Q}}}{\mathcal{B}(\mathbb{Q})} \mathcal{S}(\mathrm{O}_1[\hbar] + \mathrm{O}_2[\hbar] - g), \quad (7) $$

taking the inverse of ST to both sides of the Eq.(7),

$$ \hbar = \mathcal{S}^{-1}\left(\frac{1-\mathbb{Q}+\mathbb{Q}s^{\mathbb{Q}}}{\mathcal{B}(\mathbb{Q})} \mathcal{S}(g) \right) - \mathcal{S}^{-1}\left(\frac{1-\mathbb{Q}+\mathbb{Q}s^{\mathbb{Q}}}{\mathcal{B}(\mathbb{Q})} \mathcal{S}(\mathrm{O}_1[\hbar] + \mathrm{O}_2[\hbar]) \right), \quad (8) $$

by applying homotopy permutation method,

$$\hbar(z,\xi) = \sum_{n=0}^{\infty} \wp^n \hbar_n(z,\xi),$$

$$O_2[s(z,\xi)] = \sum_{n=0}^{\infty} \wp^n \mathcal{H}_n(\hbar), (9)$$

where

$$\mathcal{H}_n(\hbar_1,\hbar_2,\hbar_3,\ldots,\hbar_n) = \frac{1}{n!}\frac{\partial^n}{\partial \wp^n}[O_2(\sum_{i=0}^{n} \wp^i \hbar_i(z,\xi))]_{\wp=0} \quad n = 0,1,2,\ldots. \qquad (10)$$

Substituting Eq.(9) in Eq.(8), we get

$$\sum_{n=0}^{\infty} \wp^n \hbar_n(z,\xi) = G(z,\xi) - \wp\left(S^{-1}\left(\frac{1-\mathbb{Q}+\mathbb{Q}s^{\mathbb{Q}}}{\mathcal{B}(\mathbb{Q})} S\left(\sum_{n=0}^{\infty} \wp^n \mathcal{L}[\hbar_n] + \sum_{n=0}^{\infty} \wp^n \mathcal{H}_n(\hbar)\right)\right)\right), (11)$$

where, $G(z,\xi) = S^{-1}\left(\frac{1-\mathbb{Q}+\mathbb{Q}s^{\mathbb{Q}}}{\mathcal{B}(\mathbb{Q})} S(g)\right)$

By comparing both sides of Eq.(11), we can get the result below

$$\wp^0: \hbar_0(z,\xi) = \hbar_0(z) + G(z,\xi)$$

$$\wp^1: \hbar_1(z,\xi) = -S^{-1}\left(\frac{1-\mathbb{Q}+\mathbb{Q}s^{\mathbb{Q}}}{\mathcal{B}(\mathbb{Q})} S(\mathcal{L}[\hbar_0] + \mathcal{H}_0(\hbar))\right)$$

$$\vdots$$

$$\wp^n: \hbar_{n+1}(z,\xi) = -S^{-1}\left(\frac{1-\mathbb{Q}+\mathbb{Q}s^{\mathbb{Q}}}{\mathcal{B}(\mathbb{Q})} S(\mathcal{L}[\hbar_n] + \mathcal{H}_n(\hbar))\right), \quad (12)$$

We extend the solution in the following manner using the element $\wp$:

$$\hbar(z,\xi) = \sum_{n=0}^{\infty} \wp^n \hbar_n(z,\xi). \quad (13)$$

Setting $\wp = 1$ in Eq.(12), hence, we get the approximate solution of Eq.(5)

$$\hbar(z,\xi) = \lim_{\wp\to 1} \sum_{n=0}^{\infty} \wp^n \hbar_n(z,\xi) = \sum_{n=0}^{\infty} \hbar_n(z,\xi) \quad (14)$$

IV. APPLICATION

In this part, we solve linear and nonlinear differential equations with Atanagana-Baleanu operator in Reimann sense at $\mathcal{B}(\mathbb{Q}) = 1$.

Exa. 1 Let us assume the linear differential equation with ABRO and $0 < \mathbb{Q} \le 1$

$$^{ABR}\mathcal{D}_{\xi}^{\mathbb{Q}}\, \hbar - \hbar_{zz} + \hbar = 0\,, (15)$$

with the initial condition $\hbar(z,0) = z + e^{-z}$.

By applying algorithm of Sumuduhomotopy permutation method on Eq.(15), we can obtain the relationship below

$$\sum_{n=0}^{\infty} \wp^n \hbar_n = -\wp S^{-1}\left[(1-\mathbb{Q} + \mathbb{Q}s^{\mathbb{Q}}) S\left[-\sum_{n=0}^{\infty} \wp^n \hbar_{nzz} + \sum_{n=0}^{\infty} \wp^n \hbar_n\right]\right], \quad (16)$$

comparing both sides of the Eq.(16), it gives the following result

$$\wp^0: \hbar_0(z,\xi) = \hbar(z,0),$$

$$p^1: \hbar_1(z,\xi) = -\mathcal{S}^{-1}\Big[(1-\omega+\omega s^{\omega})\mathcal{S}[-p^0\hbar_{0zz} + p^0\hbar_0]\Big],$$

$$p^2: \hbar_2(z,\xi) = -\mathcal{S}^{-1}\Big[(1-\omega+\omega s^{\omega})\mathcal{S}[-p^0\hbar_{0zz} + p^0\hbar_0]\Big],$$

by subtitling the initial condition and taking ST and inverse of ST, we obtain

$$\hbar_0 = z + e^{-z},$$

$$\hbar_1 = -z\left(1-\omega+\omega\frac{\xi^{\omega}}{\Gamma(\omega+1)}\right),$$

$$\hbar_2 = z\left((1-2\omega+\omega^2)+(2\omega-2\omega^2)\frac{\xi^{\omega}}{\Gamma(\omega+1)} + \omega^2\frac{\xi^{2\omega}}{\Gamma(2\omega+1)}\right),$$

and so on.

Hence, the series of approximate solution $\hbar(z,\xi)$of Eq. (15) is

$$\hbar(z,\xi) = z\left[1-\left(1-\omega+\omega\frac{\xi^{\omega}}{\Gamma(\omega+1)}\right) + \left(\begin{array}{c}(2-4\omega+2\omega^2)\\ +(4\omega-4\omega^2)\frac{\xi^{\omega}}{\Gamma(\omega+1)}+2\omega^2\frac{\xi^{2\omega}}{\Gamma(2\omega+1)}\end{array}\right) + \cdots\right], \quad (17)$$

when we put $\omega = 1$in Eq.(17), We get the solution of Eq.(15) both roughly and precisely.

$$\hbar(z,\xi) = e^{-z} + z\left(1-\frac{\xi}{1!}+\frac{\xi^2}{2!}-\cdots\right) = e^{-z} + ze^{-\xi}.$$

Table 1: these values appear the approximate solution and exact solution of Eq.(15) for different z,ξ and ω.

z	ξ	$\hbar_{\omega=0.8}$	$\hbar_{\omega=0.9}$	$\hbar_{\omega=1}$	$\hbar$	$\lvert\hbar-\hbar_{0.8}\rvert$	$\lvert\hbar-\hbar_{0.9}\rvert$	$\lvert\hbar-\hbar_1\rvert$
0.0444	0.0444	0.9921	0.9950	0.9990	0.9990	0.0069	0.0040	0.0000
0.0879	0.0879	0.9840	0.9890	0.9964	0.9964	0.0123	0.0074	0.0000
0.1313	0.1313	0.9762	0.9823	0.9921	0.9921	0.0159	0.0098	0.0000
0.1747	0.1747	0.9690	0.9751	0.9865	0.9864	0.0174	0.0113	0.0001
0.2182	0.2182	0.9625	0.9678	0.9798	0.9794	0.0169	0.0116	0.0004
0.2616	0.2616	0.9571	0.9606	0.9719	0.9712	0.0141	0.0106	0.0007
0.3050	0.3050	0.9530	0.9536	0.9633	0.9619	0.0090	0.0083	0.0013
0.3485	0.3485	0.9501	0.9471	0.9540	0.9517	0.0016	0.0046	0.0023
0.3919	0.3919	0.9487	0.9412	0.9442	0.9406	0.0081	0.0006	0.0036
0.4353	0.4353	0.9489	0.9361	0.9341	0.9287	0.0202	0.0074	0.0054
0.4788	0.4788	0.9508	0.9319	0.9240	0.9162	0.0346	0.0158	0.0078
0.5222	0.5222	0.9545	0.9289	0.9139	0.9030	0.0515	0.0259	0.0109
0.5657	0.5657	0.9601	0.9272	0.9042	0.8893	0.0708	0.0379	0.0149
0.6091	0.6091	0.9677	0.9268	0.8949	0.8751	0.0926	0.0517	0.0198
0.6525	0.6525	0.9774	0.9281	0.8864	0.8605	0.1169	0.0676	0.0259

Exa. 2 Let's assume nonlinear differential equation with ABC operator and $0 < \wp \leq 1$

$$ {}^{ABR}D_{\xi}^{\wp}\hbar(z,\xi) - \hbar_{zz} + \hbar_z - \hbar\hbar_{zz} + \hbar^2 - \hbar = 0, \qquad (19)$$

with the initial condition $\hbar(z,0) = e^z$.

Using the algorithm of Sumuduhomotopy permutation method on Eq.(19), the equation becomes

$$\sum_{n=0}^{\infty} p^n \hbar_n = -pS^{-1}\left[(1-\wp + \wp s^{\wp})S\left[-\sum_{n=0}^{\infty} p^n \hbar_{nzz} + \sum_{n=0}^{\infty} p^n \hbar_{nz} + \sum_{n=0}^{\infty} p^n \mathcal{A}_n - \sum_{n=0}^{\infty} p^n \hbar_n\right]\right]. \qquad (20)$$

When comparing the two sides of Eq. (20), the next result is reached:

$$p^0: \hbar_0 = e^z,$$

$$p^1: \hbar_1 = -S^{-1}\left[(1-\wp+\wp s^{\wp})S[-p^0\hbar_{0zz} + p^0\hbar_{0z} + p^0\mathcal{A}_0 - p^0\hbar_0]\right],$$

$$p^2: \hbar_2 = -S^{-1}\left[(1-\wp+\wp s^{\wp})S[-p^1\hbar_{1zz} + p^1\hbar_{1z} + p^1\mathcal{A}_1 - p^1\hbar_1]\right],$$

by the above algorithms,

$$\hbar_0 = e^z,$$

$$\hbar_1 = e^z\left(1 - \wp + \wp\frac{\xi^{\wp}}{\Gamma(\wp+1)}\right),$$

$$\hbar_2 = e^z\left((1-2\wp+\wp^2) + (2\wp - 2\wp^2)\frac{\xi^{\wp}}{\Gamma(\wp+1)} + \wp^2\frac{\xi^{2\wp}}{\Gamma(2\wp+1)}\right),$$

and so on.

Therefore, the series of approximate solution $\hbar(z,\xi)$of Eq.(19) is given by

$$\hbar(z,\xi) = e^z\left[(3-3\wp+\wp^2) + (3\wp - 2\wp^2)\frac{\xi^{\wp}}{\Gamma(\wp+1)} + \wp^2\frac{\xi^{2\wp}}{\Gamma(2\wp+1)} + \cdots\right]. \qquad (21)$$

when we put $\wp = 1$in Eq.(21) , We get the solution of Eq.(19) both roughly and precisely

$$\hbar(z,\xi) = e^z\left(1 + \frac{\xi}{1!} + \frac{\xi^2}{2!} + \cdots\right) = e^{z+\xi}. \qquad (22)$$

Table 2: these values appear the approximate solution and exact solution of Eq.(15) for different z, ξ and ω.

z	ξ	$\hbar_{\omega=0.8}$	$\hbar_{\omega=0.9}$	$\hbar_{\omega=1}$	$\hbar$	$\lvert\hbar-\hbar_{0.8}\rvert$	$\lvert\hbar-\hbar_{0.9}\rvert$	$\lvert\hbar-\hbar_{1}\rvert$
0.0444	0.0444	1.4037	1.2335	1.0929	1.0929	0.3107	0.1406	0.0000
0.0879	0.0879	1.5515	1.3560	1.1920	1.1921	0.3594	0.1638	0.0001
0.1313	0.1313	1.7040	1.4860	1.2999	1.3003	0.4037	0.1857	0.0004
0.1747	0.1747	1.8642	1.6251	1.4172	1.4183	0.4459	0.2067	0.0011
0.2182	0.2182	2.0335	1.7742	1.5448	1.5470	0.4865	0.2272	0.0023
0.2616	0.2616	2.2132	1.9344	1.6833	1.6874	0.5257	0.2470	0.0041
0.3050	0.3050	2.4042	2.1066	1.8337	1.8406	0.5636	0.2660	0.0069
0.3485	0.3485	2.6075	2.2915	1.9967	2.0076	0.5999	0.2839	0.0109
0.3919	0.3919	2.8241	2.4902	2.1734	2.1898	0.6342	0.3004	0.0164
0.4353	0.4353	3.0548	2.7037	2.3648	2.3886	0.6662	0.3151	0.0238
0.4788	0.4788	3.3007	2.9329	2.5719	2.6053	0.6953	0.3276	0.0334
0.5222	0.5222	3.5627	3.1790	2.7960	2.8418	0.7209	0.3373	0.0458
0.5657	0.5657	3.8420	3.4432	3.0381	3.0997	0.7423	0.3435	0.0616
0.6091	0.6091	4.1396	3.7265	3.2998	3.3810	0.7586	0.3455	0.0812
0.6525	0.6525	4.4566	4.0304	3.5823	3.6878	0.7688	0.3425	0.1056
0.6960	0.6960	4.7944	4.3562	3.8872	4.0225	0.7719	0.3336	0.1354

V. CONCLUSIONS

The Sumuduhomotopy permutation technique was discussed in this paper and the following outcomes were attained::

The approach, which uses to solve fractional differential equations with the Atangana-Baleanu-Reimann operator, is successful and efficient.

When ω =1, the approximations solutions by this technique are close to the exact solutions.

The approach works for both linear and nonlinear equations.

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