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# The Alpha Power Exponentiated Inverse Exponential distribution and its application on Italy's COVID-19 Mortality rate data

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# Abstract:

This paper major goal is to create a model that may be used to model Italy's Covid-19 Mortality rate. The Alpha Power Exponentiated Inverse Exponential distribution, often known as the APEIEx distribution, is employed in this situation with two shapes and one or more scale parameters. In this work, significant statistical features including the Survival function, Hazard function, Quantile function, the Middle value (Median), the Lower ($1^{st}$) quantile, the Upper ($3^{rd}$) quantile, the $r^{th}$ Moment, the Moment generating function, and the order statistics are investigated. With the aid of the BFGS method, the parameters of the proposed distribution are determined using maximum likelihood estimation. When compared to other distributions employed in this investigation, the proposed distribution clearly offers a better fit for the Italy's Covid-19 Mortality rates data. This demonstrates the flexibility and adaptability of the APEIEx distribution for the Covid-19 Mortality rates in Italy.

***Keywords-*** Alpha Power, Exponentiated Inverse Exponential distribution, Mortality rate, Maximum Likelihood estimation.

## I. INTRODUCTION

The concept of statistics (probability density function) is useful in various fields of study like medicine, engineering, economics, and others. In real-life analysis, many data sets display characteristics of a bathtub, an inverted bathtub, skewness, kurtosis, monotonic increase, monotonic decrease, and many more. Scholars may find it difficult to find a suitable model for modeling such data sets. For this reason, statisticians bring up the technique of model modification to help solve real-life phenomena.

Data sets with inverted bathtub failure rates can be modeled using the Inverse Exponential (IEx) distribution; however, data sets with heavy tails or high skewness cannot be well modeled. Additionally, the IEx distribution is unsuitable for modeling data with bimodality and a high bathtub failure rate. Due to this outcome of the inverted Exponential distribution, many scholars have taken up the responsibility to extend or add parameters to the inverted Exponential distribution to make it more flexible and adaptable to solve real-life phenomena.

Marshall-Olkin Alpha Power Inverse Exponential (MOAPIE) Distribution: Properties and Applications by [1] and applied the proposed new distribution to real data representing the survival times in days of guinea pigs injected with different doses of tubercle bacilli is given and its goodness-of-fit is demonstrated. On the

Exponentiated Generalized Inverse Exponential (EGIE) Distribution by [2], and find out that the model can be successfully used to model lifetime data sets and real-life phenomena with inverted bathtub failure rates. The flexibility of the Transmuted Inverse Exponential (TIE) Distribution by [3], demonstrates that the TIE distribution is more robust than the Inverse Exponential distribution. Properties and Applications of a Two-Parameter Inverse Exponential (IE) Distribution with a Decreasing Failure Rate by [4]. Statistical Properties of the Exponentiated Generalized Inverted Exponential Distribution by [5]. The Gompertz Inverse Exponential (GoIE) distribution with applications by [6]. The Inverse Weibull Inverse Exponential (IWIE) distribution with Application by [7]. The Transmuted Inverse Exponential (TIE) distribution by [8]. Theoretical Analysis of the Kumaraswamy-Inverse Exponential (K-IE) distribution by [9]. The Exponentiated Inverted Exponential (EIEx) distribution by [10].

If $X$ is a non-negative exponential random variable, then the distribution of a random variable $Z = 1/X$, defines an IEx distribution. [11], defines the Inverse Exponential (IEx) distribution with CDF, $F(z)$ and PDF, $f(z)$ respectively, as follows;

$$F_{IEx}(z) = e^{-\frac{\eta}{z}} \tag{1}$$

The corresponding eq. for eq. (1) is given as;
$f_{IEx}(z) = \frac{d}{dz}\Big(F_{IEx}(z)\Big)$

$$f_{IEx}(z) = \frac{\eta}{z^2}e^{-\frac{\eta}{z}} \tag{2}$$

$\forall z, \eta > 0$ and $\eta$ is a scale parameter.

The Alpha Power family has been used by various scholars in the field of statistics to extend well-known traditional distributions.

The Extended Alpha Power Transformed Family of Distributions: Properties and Applications by [12]. The Alpha Power Transformation Family: Properties and Applications by [13]. The Censored Beta-Skew Alpha Power Distribution by [14]. Alpha Power Exponentiated Inverse Rayleigh distribution and its applications to real and simulated data by [15]. Alpha Power Transformed Weibull-G Family of Distributions: Theory and Applications by [16]. Alpha Power Inverted Exponential Distribution: Properties and Application by [17]. A new alpha power transformed family of distributions: properties and applications to the Weibull model by [18]. A new extended alpha power transformed family of distributions: properties and applications by [19]. A new extended alpha power transformed family of distributions: properties, characterizations and an application to a data set in the insurance sciences by [20].

The Alpha Power transform is define by [21] with cumulative distribution function (CDF) as;

$$F_{APT}(z) = \frac{\alpha^{F(z)} - 1}{\alpha - 1} \tag{3}$$

The corresponding eq. to eq. (3) is;

$$f_{APT}(z) = \frac{\ln(\alpha)}{\alpha - 1} f(z)\alpha^{F(z)} \tag{4}$$

$\forall z, \alpha > 0, \alpha \neq 1$ and $\alpha$ is a shape parameter.

Other families of distributions have been used by scholars to extend some well-known distributions, like the Exponential, Rayleigh, and Lindley distributions. For further reading, scholars can read from [22], [23], [24], [25], [26], and [27] to name but a few.

# II. ALPHA POWER EXPONENTIATED INVERSE EXPONENTIAL (APEIEx) DISTRIBUTION

To develop the APEIEx distribution, we need the help of the Exponentiated Inverse Exponential (EIEx) distribution.

## A. *The EIEx distribution*

The EIEx distribution CDF can be derived by exponentiating eq. (1) to a certain constant.

$$\begin{aligned} F(z) &= \Big(F_{IEx}(z)\Big)^{\beta} \\ &= \Big(e^{-\frac{\eta}{z}}\Big)^{\beta} \end{aligned} \tag{5}$$

From eq. (5), the PDF is; $f(z) = \frac{d}{dz}\Big(F(z)\Big)$

$$f(z) = \frac{\beta\eta}{z^2}\Big(e^{-\frac{\eta}{z}}\Big)^{\beta} \tag{6}$$

$\forall z, \eta, \beta > 0$ and $\beta$ is a shape parameter.

## B. *The APEIEx distribution*

The APEIEx distribution can be derive by utilizing eq. (5) into eq. (3) and eqs. (5) & (6) into eq. (4).

$$\text{CDF; } F_{APEIEx}(z) = \frac{\alpha^{e^{-\frac{\eta\beta}{z}}} - 1}{\alpha - 1} \tag{7}$$

$$\text{PDF; } f_{APEIEx}(z) = \beta\eta z^{-2}\frac{\ln(\alpha)}{\alpha - 1}e^{-\frac{\eta\beta}{z}}\alpha^{e^{-\frac{\eta\beta}{z}}} \tag{8}$$

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$\forall z, \alpha, \beta > 0, \alpha \neq 1$ and $\alpha\beta$ are shape parameters & $\eta$ is a scale parameter.

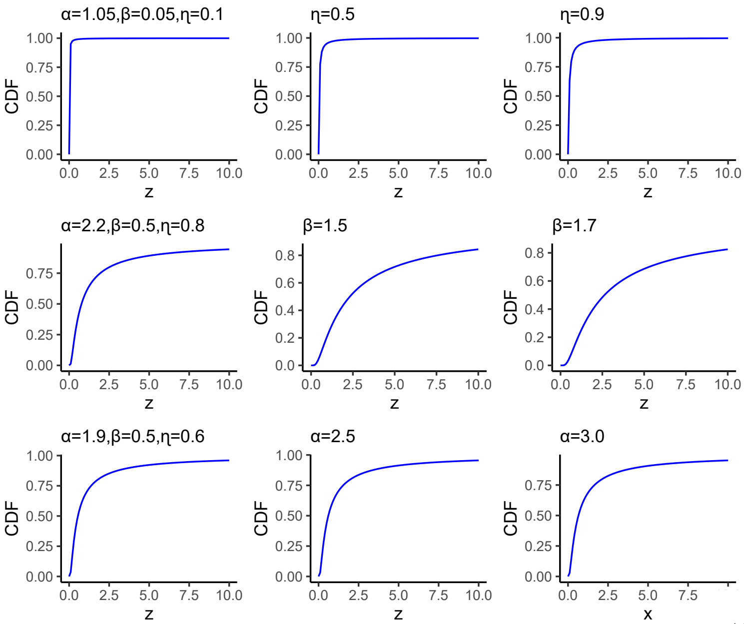


**Figure 1: The CDF of the APEIEx distribution.**

Figure 1, displays the graphical shapes of the CDF of APEIEx for selected parameter values, and the shapes are monotonic increase or non-monotonic decrease.

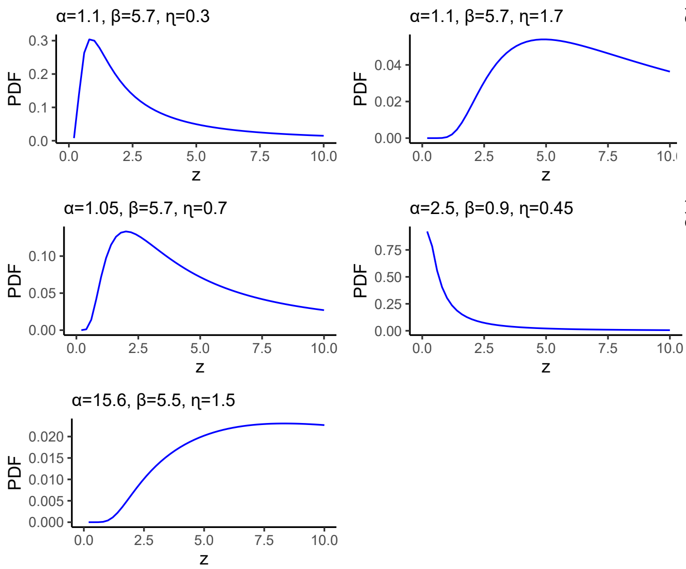


**Figure 2: The PDF of the APEIEx distribution.**

Figure 2, displays the graphical shapes of the PDF of APEIEx distribution for selected parameter values, and the shapes are reverse J-shape, positively skewed, increasing, left-skewed, and unimodal.

The hazard and survival functions in relation to the probability density function are given as follows:

$$h_{APEIEx}(z) = \frac{\beta\eta z^{-2}\ln(\alpha) e^{-\frac{\eta\beta}{z}} \alpha^{e^{-\frac{\eta\beta}{z}}}}{\alpha - \alpha^{e^{-\frac{\eta\beta}{z}}}} \tag{9}$$

$$S_{APEIEx}(z) = \frac{\alpha - \alpha^{e^{-\frac{\eta\beta}{z}}}}{\alpha - 1} \tag{10}$$

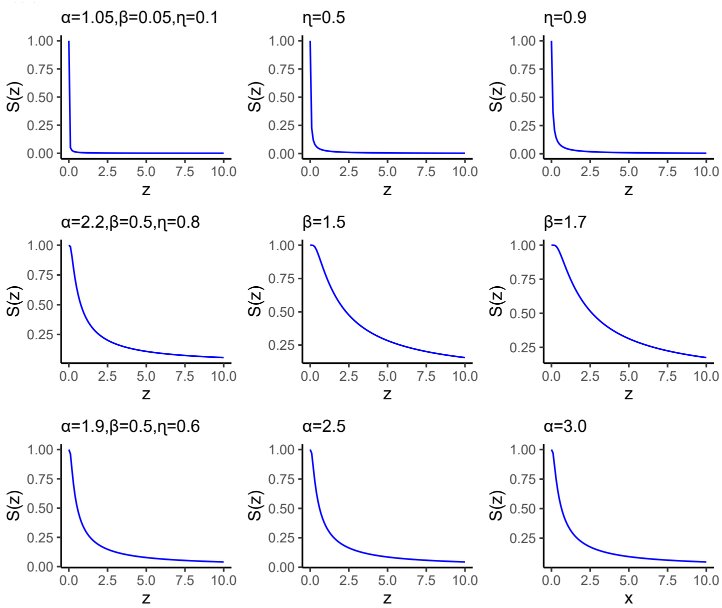


**Figure 3: The $S_{APEIEx}(z)$ of the APEIEx distribution.**

Figure 3, displays the graphical shapes of the $S_{APEIEx}(z)$ of APEIEx distribution for selected parameter values and the shapes shows the reverse of the CDF shapes (i.e. monotonic decrease or non-monotonic increase).

Figure 4, displays the graphical shapes of the $h_{APEIEx}(z)$ of the APEIEx distribution for selected parameter values and the shapes of increasing, decreasing, an inverted bathtub.

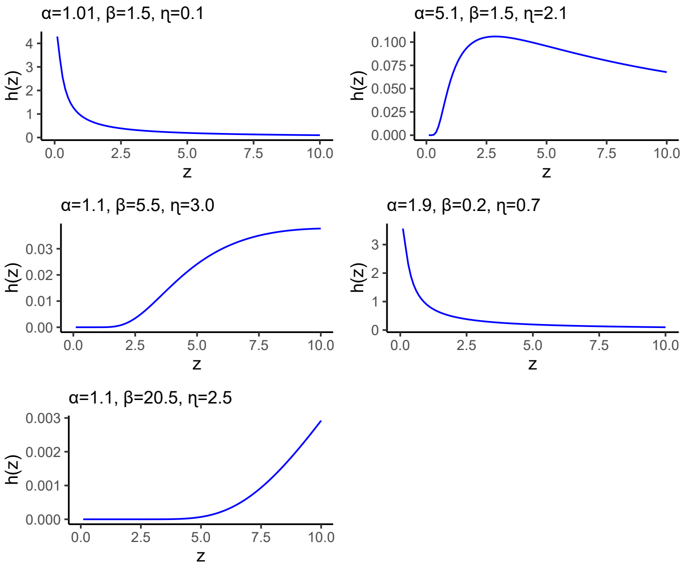


**Figure 4: The $h_{APEIEx}(z)$ of the APEIEx distribution.**

# III. IMPORTANT STATISTICAL PROPERTIES

In this section, all the important statistical properties are derived.

## A. *Quantile function*

The random variable of the APEIEx can be calculated by solving for $z$ from eq. (7).

$$\text{Let } \lambda = F_{APEIEx}(z) \qquad z = F^{-1}_{APEIEx}(\lambda) \tag{11}$$

Hence the random variable $z_\lambda$ for $\lambda \in (0,1)$ is given as;

$$z_\lambda = \frac{-\eta\beta}{\ln\left\{\frac{\ln\left(1+\lambda(\alpha-1)\right)}{\ln(\alpha)}\right\}} \tag{12}$$

The median of the APEIEx is by putting $\lambda = 1/2$, and we have;

$$z_{1/2} = \frac{-\eta\beta}{\ln\left\{\frac{\ln\left(1/2+\alpha/2\right)}{\ln(\alpha)}\right\}} \tag{13}$$

Lower ($1^{st}$) Quantile of the APEIEx distribution is by putting $\lambda = 1/4$, and we have;

$$z_{1/4} = \frac{-\eta\beta}{\ln\left\{\frac{\ln\left(3/4+\alpha/4\right)}{\ln(\alpha)}\right\}} \tag{14}$$

**Table 1: Quantile Values for APEIEx($\alpha, \beta, \eta$) distribution.**

| $\lambda$ | (1.5, 0.05, 0.9) | (1.6, 0.05, 0.8) | (1.7,0.06,0.7) |
|---|---|---|---|
| 0.1 | 0.0213 | 0.0192 | 0.0204 |
| 0.2 | 0.0311 | 0.0281 | 0.0300 |
| 0.3 | 0.0422 | 0.0383 | 0.0410 |
| 0.4 | 0.0563 | 0.0512 | 0.0549 |
| **0.5** | **0.0753** | **0.0686** | **0.0737** |
| 0.6 | 0.1034 | 0.0943 | 0.1014 |
| 0.7 | 0.1495 | 0.1366 | 0.1470 |
| 0.8 | 0.2413 | 0.2205 | 0.2377 |
| 0.9 | 0.5154 | 0.4716 | 0.5087 |

The Upper ($3^{rd}$) Quantile of the APEIEx distribution is by putting $\lambda = 3/4$, and we have;

$$z_{3/4} = \frac{-\eta\beta}{\ln\left\{\frac{\ln\left(1/4+3\alpha/4\right)}{\ln(\alpha)}\right\}} \tag{15}$$

## B. *Moment*

For $Z \sim$APEIEx($\alpha$,$\lambda$,$\eta$), the $r^{th}$ moment can express as;

$$\omega'_r = E(Z^r) = \int_0^\infty z^r \beta\eta z^{-2} \frac{\ln(\alpha)}{\alpha-1} e^{-\frac{\eta\beta}{z}} \alpha^{e^{-\frac{\eta\beta}{z}}} dz \tag{16}$$

Using series notation $\alpha^x = \sum_{k=0}^{\infty} \frac{(\ln\alpha)^k}{k!} x^k$

$$\alpha^{e^{-\frac{\eta\beta}{z}}} = \sum_{k=0}^{\infty} \frac{(\ln\alpha)^k}{k!} \left(e^{-\frac{\eta\beta}{z}}\right)^k$$

$$\begin{aligned}\omega'_r &= \frac{\beta\eta\ln(\alpha)}{\alpha-1} \int_0^\infty z^r z^{-2} e^{-\frac{\eta\beta}{z}} \sum_{k=0}^{\infty} \frac{(\ln\alpha)^k}{k!} \left(e^{-\frac{\eta\beta}{z}}\right)^k dz \\ &= \frac{\beta\eta\ln(\alpha)}{\alpha-1} \sum_{k=0}^{\infty} \frac{(\ln\alpha)^k}{k!} \int_0^\infty z^{r-2} \left(e^{-\frac{\eta\beta}{z}}\right)^{k+1} dz\end{aligned} \tag{17}$$

### *Skewness and Kurtosis*

In relation of the APEIEx distribution, the Galton skewness and Moors kurtosis is define as;

$$G_s = \frac{Q(1/4) + Q(3/4) - 2Q(1/2)}{Q(3/4) - Q(1/4)}$$

$$M_k = \frac{Q(3/8) + Q(7/8) - Q(1/8) - Q(2/8)}{Q(6/8) - Q(2/8)}$$

where $Q$ = quantile values

The effects of the additional parameters $\alpha\&\beta$ is seen clearly in Figures 5 and Figures 6.

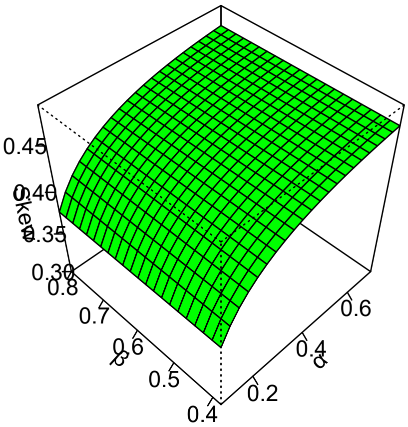


**Figure 5: The Skewness values for APEIEx distribution.**

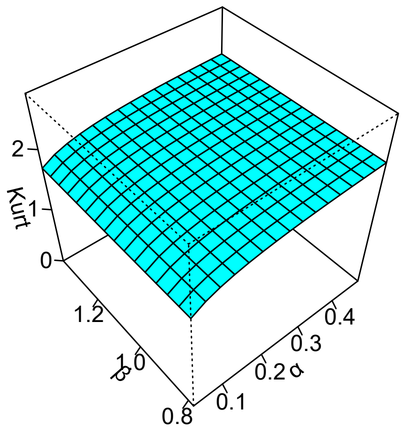


**Figure 6: The Kurtosis values for APEIEx distribution.**

## C. *Moment generating function (Mgf)*

For $Z \sim$APEIEx($\alpha$,$\lambda$,$\eta$), the Mgf can express as;

$$M_Z(t) = E(e^{tZ}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(Z^r) \tag{18}$$

Utilizing $E(Z^r)$ from eq. (17)

$$M_Z(t) = \frac{\beta\eta\ln(\alpha)}{\alpha-1} \sum_{r=0}^{\infty}\sum_{k=0}^{\infty} \frac{t^r}{r!}\frac{(\ln\alpha)^k}{k!} \int_0^{\infty} z^{r-2}\left(e^{-\frac{\eta\beta}{z}}\right)^{k+1} dz \tag{19}$$

## D. *Order Statistics*

For the ordered random sample $Z_1, Z_2, Z_3, \ldots, Z_n$, the $k^{th}$ minimum and maximum order statistics of the APEIEx distribution are given as follows;

$$f_{Z_1}(z) = n\sum_{t=0}^{n-1}(-1)^t\binom{n-l}{t} f_{APEIEx}(z)\Big(F_{APEIEx}(z)\Big)^t$$

$$f_{Z_n}(z) = n f_{APEIEx}(z)\Big(F_{APEIEx}(z)\Big)^{n-1} \tag{20}$$

Hence;

$$f_{Z_1}(z) = n\beta\eta z^{-2}\frac{\ln(\alpha)}{\alpha-1}e^{-\frac{\eta\beta}{z}}\alpha^{e^{-\frac{\eta\beta}{z}}}\sum_{t=0}^{n-1}(-1)^t\binom{n-l}{t}\times \left(\frac{\alpha^{e^{-\frac{\eta\beta}{z}}}-1}{\alpha-1}\right)^t$$

$$f_{Z_n}(z) = n\beta\eta z^{-2}\frac{\ln(\alpha)}{\alpha-1}e^{-\frac{\eta\beta}{z}}\alpha^{e^{-\frac{\eta\beta}{z}}}\left(\frac{\alpha^{e^{-\frac{\eta\beta}{z}}}-1}{\alpha-1}\right)^{n-1} \tag{21}$$

# IV. PARAMETER ESTIMATIONS

Since n is the random sample from APEIEx$\sim(\alpha,\beta,\eta)$, the joint probability density function is;

$$\begin{aligned}\ln L(\theta) &= \ln\left(\prod_{i=1}^{n} f(x_i;\theta)\right) \quad ;\text{where} \quad \theta\in(\alpha,\beta,\eta)>0 \\ &= n\ln(\beta)+n\ln(\eta)-n\ln(\alpha-1)+n\ln(\ln\alpha) \\ &\quad -2\sum_{i=1}^{n}\ln(z_i)-\eta\beta\sum_{i=1}^{n}z_i^{-1}+\ln(\alpha)\sum_{i=1}^{n}e^{-\frac{\eta\beta}{z_i}}\end{aligned} \tag{22}$$

From eq. (22), differentiate wrt $\alpha, \beta, \eta$ and equate the results to zero;

$$\frac{\partial\ln L(\theta)}{\partial\beta} = \frac{n}{\beta} - \eta\sum_{i=1}^{n}z_i^{-1} - \eta\ln(\alpha)\sum_{i=1}^{n}\left(z_i^{-1}e^{-\frac{\eta\beta}{z_i}}\right) = 0 \tag{23}$$

$$\frac{\partial\ln L(\theta)}{\partial\eta} = \frac{n}{\eta} - \beta\sum_{i=1}^{n}z_i^{-1} - \beta\ln(\alpha)\sum_{i=1}^{n}\left(z_i^{-1}e^{-\frac{\eta\beta}{z_i}}\right) = 0 \tag{24}$$

$$\frac{\partial\ln L(\theta)}{\partial\alpha} = \frac{n}{\alpha\ln(\alpha)} - \frac{n}{\alpha-1} + \frac{1}{\alpha}\sum_{i=1}^{n}e^{-\frac{\eta\beta}{z_i}} = 0 \tag{25}$$

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From eq. (23) to eq. (25) are non-linear, and their numerical solutions are needed.
Here we will use the BFGS algorithm, and both the gradient vector of the log-likelihood function and the Hessian matrix are needed. Hence, the observed information matrix is;

$$J^{-1}(\lambda) = \left\{ \begin{array}{ccc} \frac{\partial^2 \ln L(\theta)}{\partial \alpha^2} & \frac{\partial^2 \ln L(\theta)}{\partial \alpha \partial \beta} & \frac{\partial^2 \ln L(\theta)}{\partial \alpha \partial \eta} \\ & \frac{\partial^2 \ln L(\theta)}{\partial \beta^2} & \frac{\partial^2 \ln L(\theta)}{\partial \beta \partial \eta} \\ & & \frac{\partial^2 \ln L(\theta)}{\partial \eta^2} \end{array} \right\}$$

where $\lambda = (\alpha, \beta, \eta)'$.
The expressions for terms in the Hessian matrix are available if need arises.

For large sample $(1-\delta)100\%$ confidence interval for the APEIEx parameters are express as follows;

$$\widehat{\alpha} \pm Z_{\delta/2}\sqrt{\Sigma_{11}}$$

$$\widehat{\beta} \pm Z_{\delta/2}\sqrt{\Sigma_{22}}$$

$$\widehat{\eta} \pm Z_{\delta/2}\sqrt{\Sigma_{33}}$$

# V. MONTE CARLO SIMULATION

Through simulation with 1000 repetitions for each sample size (n) = 75,125,175,...,425 the parameter estimates of APEIEx distribution, the Mean Square Error (MSE), Average Bias (AB), and Root Mean Square Error (RMSE) are calculated.
The random samples used for the simulation are calculated through the help of eq. (12)

$$z_\lambda = \frac{-\eta\beta}{\ln\left\{\frac{\ln\left(1+\lambda(\alpha-1)\right)}{\ln(\alpha)}\right\}} \tag{26}$$

where $\lambda \in (0,1)$
The equation Mean Square Error (MSE), Average Bias (AB), and Root Mean Square Error (RMSE) are define respectively as;

$$\text{MSE} = \frac{1}{M}\sum_{i=1}^{M}(\widehat{\Phi}_i - \Phi)^2 \tag{27}$$

$$\text{AB} = \frac{1}{M}\sum_{i=1}^{M}(\widehat{\Phi}_i - \Phi) \tag{28}$$

and

$$\text{RMSE} = \sqrt{\frac{1}{M}\sum_{i=1}^{M}(\widehat{\Phi}_i - \Phi)^2} \tag{29}$$

**Table 2: The estimates, corresponding ABs, MSEs, and RMSEs.**

| $\alpha = 0.50, \beta = 1.50, \eta = 2.00$ | | | | | | |
|---|---|---|---|---|---|---|
| n | **Estimates** | | | **ABs** | | |
| | $\widehat{\alpha}$ | $\widehat{\beta}$ | $\widehat{\eta}$ | $\widehat{\alpha}$ | $\widehat{\beta}$ | $\widehat{\eta}$ |
| 75 | 0.6005 | 1.5308 | 2.0158 | 0.1005 | 0.0308 | 0.0158 |
| 125 | 0.5821 | 1.5105 | 2.0000 | 0.0820 | 0.0105 | 0.000039 |
| 175 | 0.5609 | 1.5028 | 1.9969 | 0.0609 | 0.0028 | -0.0031 |
| 225 | 0.5236 | 1.5104 | 2.0033 | 0.0236 | 0.0104 | 0.0033 |
| 275 | 0.5236 | 1.5078 | 2.0021 | 0.0236 | 0.0078 | 0.0021 |
| 325 | 0.5234 | 1.5058 | 2.0008 | 0.0234 | 0.0058 | 0.00077 |
| 375 | 0.5169 | 1.5047 | 2.0004 | 0.0169 | 0.0047 | 0.00036 |
| 425 | 0.5119 | 1.5095 | 2.0044 | 0.0119 | 0.0095 | 0.0044 |
| 475 | 0.5118 | 1.5040 | 2.0003 | 0.0118 | 0.0040 | 0.00025 |
| n | **MSEs** | | | **RMSEs** | | |
| | $\widehat{\alpha}$ | $\widehat{\beta}$ | $\widehat{\eta}$ | $\widehat{\alpha}$ | $\widehat{\beta}$ | $\widehat{\eta}$ |
| 75 | 0.5020 | 0.0313 | 0.0228 | 0.7085 | 0.1769 | 0.1511 |
| 125 | 0.1623 | 0.0194 | 0.0138 | 0.4029 | 0.1392 | 0.1174 |
| 175 | 0.0925 | 0.0127 | 0.0084 | 0.3042 | 0.1127 | 0.0918 |
| 225 | 0.0550 | 0.0109 | 0.0075 | 0.2346 | 0.1045 | 0.0864 |
| 275 | 0.0465 | 0.0086 | 0.0055 | 0.2156 | 0.0929 | 0.0743 |
| 325 | 0.0371 | 0.0074 | 0.0047 | 0.1926 | 0.0859 | 0.0686 |
| 375 | 0.0276 | 0.0060 | 0.0036 | 0.1662 | 0.0772 | 0.0603 |
| 425 | 0.0244 | 0.0055 | 0.0035 | 0.1563 | 0.0743 | 0.0593 |
| 475 | 0.0228 | 0.0049 | 0.0031 | 0.1511 | 0.0697 | 0.0555 |

**Table 3: The estimates, corresponding ABs, MSEs, and RMSEs.**

| $\alpha = 0.30, \beta = 1.30, \eta = 1.80$ | | | | | | |
|---|---|---|---|---|---|---|
| n | **Estimates** | | | **ABs** | | |
| | $\widehat{\alpha}$ | $\widehat{\beta}$ | $\widehat{\eta}$ | $\widehat{\alpha}$ | $\widehat{\beta}$ | $\widehat{\eta}$ |
| 75 | 0.3488 | 1.3249 | 1.8089 | 00.0488 | 0.0249 | 0.0089 |
| 125 | 0.3405 | 1.3093 | 1.7991 | 0.0405 | 0.0093 | -0.00086 |
| 175 | 0.3314 | 1.3033 | 1.7962 | 0.0314 | 0.0033 | -0.0038 |
| 225 | 0.3116 | 1.3079 | 1.8023 | 0.0116 | 0.0079 | 0.0023 |
| 275 | 0.3117 | 1.3064 | 1.8009 | 0.0117 | 0.0064 | 0.00086 |
| 325 | 0.3116 | 1.3048 | 1.8002 | 0.0116 | 0.0048 | 0.00022 |
| 375 | 0.3083 | 1.3041 | 1.7999 | 0.0083 | 0.0041 | -0.000063 |
| 425 | 0.3057 | 1.3077 | 1.8033 | 0.0057 | 0.0077 | 0.0033 |
| 475 | 0.3059 | 1.3033 | 1.7997 | 0.0059 | 0.0033 | -0.00031 |
| n | **MSEs** | | | **RMSEs** | | |
| | $\widehat{\alpha}$ | $\widehat{\beta}$ | $\widehat{\eta}$ | $\widehat{\alpha}$ | $\widehat{\beta}$ | $\widehat{\eta}$ |
| 75 | 0.1153 | 0.0204 | 0.0125 | 0.3396 | 0.1428 | 0.1117 |
| 125 | 0.0456 | 0.0122 | 0.0076 | 0.2136 | 0.1105 | 0.0874 |
| 175 | 0.0270 | 0.0078 | 0.0050 | 0.1642 | 0.0884 | 0.0706 |
| 225 | 0.0176 | 0.0070 | 0.0041 | 0.1325 | 0.0837 | 0.0642 |
| 275 | 0.0144 | 0.0054 | 0.0032 | 0.1198 | 0.0738 | 0.0562 |
| 325 | 0.0115 | 0.0046 | 0.0026 | 0.1072 | 0.0680 | 0.0512 |
| 375 | 0.0089 | 0.0037 | 0.0021 | 0.0944 | 0.0611 | 0.0453 |
| 425 | 0.0079 | 0.0034 | 0.0020 | 0.0888 | 0.0584 | 0.0446 |
| 475 | 0.0075 | 0.0030 | 0.0018 | 0.0866 | 0.0549 | 0.0422 |

where $\Phi(\alpha, \beta, \eta)$ and $M$ number of repetitions

According to Table 2 and Table 3, the estimates, average bias, mean square error, and root mean square error all tend to decrease for the chosen initial parameters as the sample size (n) rises.

# VI. APPLICATION

We used actual data sets in this part to apply the APEIEx distribution. Information criteria, goodness-of-fit, negative log-likelihood, and the p-value are used to assess the APEIEx distribution's performance.

The proposed distribution is compared with well-known distributions, these distributions are;

- Alpha Power Inverted Exponential (APIEx) distribution by [17]

$$F_{APIEx}(z) = (\alpha - 1)^{-1}\left(\alpha^{e^{-\frac{\eta}{z}}} - 1\right)$$
$$f_{APIEx}(z) = \eta \log(\alpha)(\alpha - 1)^{-1} z^{-2} e^{-\frac{\eta}{z}} \alpha^{e^{-\frac{\eta}{z}}} \quad ; z, \alpha, \eta > 0, \alpha \neq 1$$

- Generalized Alpha Power Inverted Exponential (GAPIEx) distribution

$$F_{GAPIEx}(z) = (\alpha - 1)^{-\beta}\left(\alpha^{e^{-\frac{\eta}{z}}} - 1\right)^{\beta}$$
$$f_{GAPIEx}(z) = \beta\eta \log(\alpha)(\alpha - 1)^{-\beta} z^{-2} e^{-\frac{\eta}{z}} \alpha^{e^{-\frac{\eta}{z}}} \times \left(\alpha^{e^{-\frac{\eta}{z}}} - 1\right)^{\beta-1} \quad ; z, \alpha, \eta > 0, \alpha \neq 1$$

- Exponentiated Inverse Exponential (EIEx) distribution

$$F_{EIEx}(z) = \left(e^{-\frac{\eta}{z}}\right)^{\beta}$$
$$f_{EIEx}(z) = \frac{\beta\eta}{z^2}\left(e^{-\frac{\eta}{z}}\right)^{\beta} \quad ; z, \beta, \eta > 0$$

- Exponential (Ex) distribution

$$F_{Ex}(z) = 1 - e^{-\eta z}$$
$$f_{Ex}(z) = \eta e^{-\eta z} \quad ; z, \eta > 0$$

- Inverse Exponential (IEx) distribution

$$F_{IEx}(z) = e^{-\frac{\eta}{z}}$$
$$f_{IEx}(z) = \frac{\eta}{z^2} e^{-\frac{\eta}{z}} \quad ; z, \eta > 0$$

### *Data Set: Italy's COVID-19 mortality rate data*

Table 4, consist of Italy COVID-19 mortality rates data, and the data is retrieved from [23].

The summary statistics for data is provided in Table 5. The data is positively skew and the kurtosis is <3, hence the data is platykurtic.

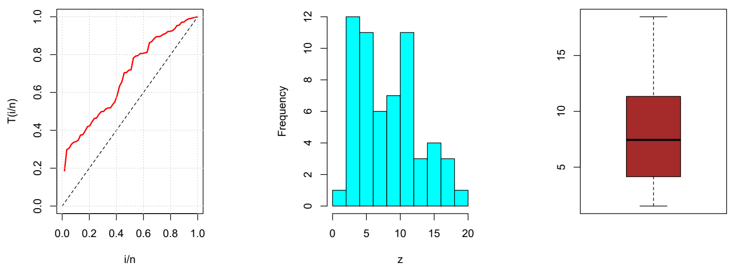
**Table 4: Italy's COVID-19 mortality rate data.**

| | | | | | | |
|---|---|---|---|---|---|---|
| 4.571 | 7.201 | 3.606 | 8.479 | 11.410 | 8.961 | 10.919 |
| 10.908 | 6.503 | 18.474 | 11.010 | 17.337 | 16.561 | 13.226 |
| 15.137 | 8.697 | 15.787 | 13.333 | 11.822 | 14.242 | 11.273 |
| 14.330 | 16.046 | 8.646 | 8.905 | 8.906 | 7.407 | 7.445 |
| 7.214 | 6.194 | 4.640 | 5.542 | 5.073 | 4.416 | 4.859 |
| 4.408 | 4.639 | 3.148 | 4.040 | 4.253 | 4.011 | 3.564 |
| 3.827 | 3.134 | 2.780 | 2.881 | 3.341 | 2.686 | 2.814 |
| 2.508 | 2.450 | 1.518 | 11.950 | 10.282 | 11.775 | 10.644 |
| 10.138 | 9.037 | 12.396 | | | | |

**Table 5: Summary Statistics of Italy COVID-19 mortality rates data.**

| N | Max. | Min. | Mean | Median | Mode | Skew | Kurt |
|---|---|---|---|---|---|---|---|
| 59 | 18.47 | 1.52 | 8.16 | 7.45 | 4.57 | 0.46 | -0.84 |

The TTT-transform plot shows a concave above the $45^0$ line and no outliers in the boxplot, as shown in Figure 7 for the data set. This gives a clear picture of how the shapes in Figure 4 (hazard function shapes) can model the data set.




(a) TTT-transform (b) Histogram (c) Boxplot

**Figure 7: The TTT-transform plot, the Boxplot and the Histogram of Italy COVID-19 mortality rates data.**

The maximum likelihood with a standard in brackets of the APEIEx distribution and other well-known distributions are provided in Table 6.

**Table 6: The MLEs and the Standard Error (in parentheses) for Italy COVID-19 mortality rates data.**

| Distribution | $\widehat{\alpha}$ | $\widehat{\beta}$ | $\widehat{\eta}$ |
|---|---|---|---|
| **APEIEx** | 0.0111(0.0116) | 1.9669(1.3402) | 6.1440(4.1865) |
| GAPIEx | 1.0072(0.4498) | 3.4118(5.5168) | 1.6518(2.1455) |
| APIEx | 0.0111(0.0115) | − | 12.0815(1.5058) |
| EIEx | − | 4.9901(55.2049) | 1.1303(12.5038) |
| Ex | − | − | 0.1226(0.0160) |
| IEx | − | − | 5.6406(0.7343) |

The APEIEx distribution fits the COVID-19 death

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rates data for Italy better than the other distributions, as seen in Table 7. The APIEx distribution also fits the data, however the APEIEx distribution offers the highest negative log-likelihood ($-\ln L$), smallest Akaike information criterion ($AIC$), lowest Kolmogorov-Smirnov test ($K-S$), and highest $p-value$.

**Table 7: The $-\ln L$, Goodness-of-fit and the $p-values$ results for Italy COVID-19 mortality rates data.**

| Distribution | $-\ln L$ | $AIC$ | $K-S$ | $p-value$ |
|---|---|---|---|---|
| **APEIEx** | **173.5015** | **353.0031** | **0.1599** | **0.0873** |
| GAPIEx | 184.1777 | 374.3555 | 0.2637 | <5% |
| APIEx | 173.5020 | 351.0040 | 0.1603 | 0.0863 |
| EIEx | 184.1642 | 372.3285 | 0.2631 | <5% |
| Ex | 182.8387 | 367.6775 | 0.2425 | <5% |
| IEx | 184.1642 | 370.3285 | 0.2631 | <5% |

Figure 8, which provides the fitted densities of Italy's COVID-19 mortality rates, makes it clear that the APEIEx and APIEx distributions fit the data well and the GAPIEx, EIEx, Ex, and IEx distributions do not fit the data.

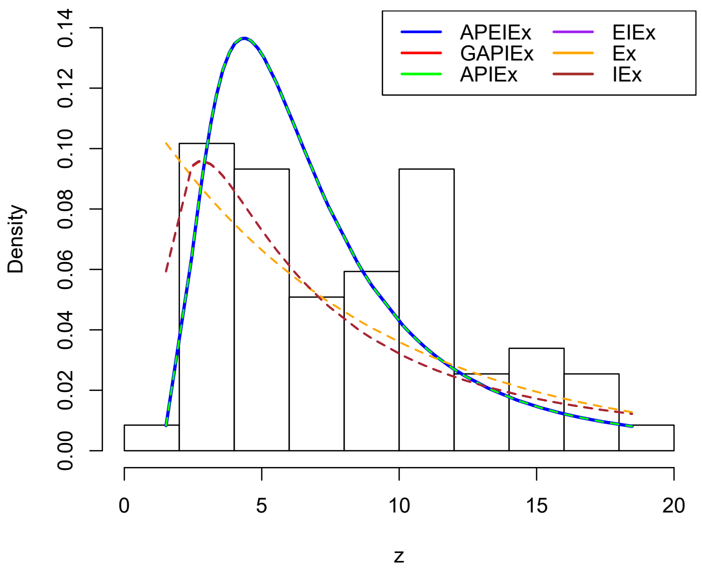


**Figure 8: The Fitted densities plot of Italy COVID-19 mortality rates data.**

***Conclusion for Italy's COVID-19 mortality rates data application***

- In terms of data application, we can see that the APEIEx distribution outperforms the other models utilized in the study in terms of negative log-likelihood, information criterion ($AIC$), and goodness-of-fit ($K-S$).
- The GAPIEx, EIEx, Ex, and IEx distributions are all <5% in terms of the $p-values$, whereas the APEIEx has the greatest value.
- The APEIEx distribution fitted the data well than the other models used, although the APIEx distribution also provide better fitted, see Figure 8.

# VII. CONCLUSION

This paper proposes a new distribution called the "Alpha Power Exponentiated Inverse Exponential (APEIEx) distribution". The main aim of this study is to develop the APEIEx distribution and use it on data sets that show characteristics of skewness, kurtosis, bathtub, and inverted bathtub. Some important statistical properties of the APEIEx are derived. The APEIEx distribution is applied to uncensored data and compared with other well-known models by using information criterion ($AIC$), goodness-of-fit ($K-S$), and their $p-values$.
The APEIEx distribution provided the best fit on the data set, with the highest negative log-likelihood ($-\ln L$) & $p-values$ and the smallest $AIC$ & $K-S$.
Furthermore, researcher(s) can use the new model on censored data for further studies, and for more flexibility, future researcher(s) can use the Transmuted technique.

## Author contribution

**Conceptualization:** Moses Kargbo
**Analysis:** Moses Kargbo
**Methodology:** Moses Kargbo
**Visualization & Software:** Moses Kargbo
**Writing-Review & Editing:** Moses Kargbo, Anthony Gichuhi Waititu, & Anthony Kibira Wanjoya

## Conflicts of Interest

All authors have read and agreed to the published version of the manuscript.

## Data availability

The real data used to illustrate the proposed distribution are within the manuscript.

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