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Independent Elements and Minimal Generating Sets in Infinite Semigroups

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Abstract:
This paper extends the paper results by (Effiong and Co, 2023) on intersection of coset of Cyclic Subsemigroup to obtain algorithm for minimal generators of semigroup. A semigroup generating set is subset of semigroups which determines the whole semigroup through algebraic closure. The minimal generating set isthat which does not have a proper generating subset. The paper also shows an idea about the infiniteness of the independent elements in an infinite semigroup. An example is given to show how relatively small the minimal generating sets of such infinite semigroups are.

Keywords **—**Set, Cosets, Semigroup, Subsemigroup, Cyclic Semigroup, Independent, Generating, Commutative Semigroup..

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1.0. INTRODUCTION AND PRELIMINARIES

(Effiong and co, 2023) gave a brief discussion on cosets of cyclic subsemigroups S and the partitioning of subsemigroups. Such partitioning results in minimal generators of semigroups. They also discussed intersection of cosets of cyclic subsemigroups, and showed with a theorem by (Sampson, 2022), the analog of a result on normal subgroups of a group for semigroups.

This paper extends that work of (Effiong and co, 2023), using some results by (Sampson, 2022) about algorithms for determining Minimal generator of a semigroup. It also gives idea about how mostly infinite the independent elements (See (Lipcsey and Sampson, 2019) and (Lipcsey and Sampson, 2023) for explanation of the concept of independence) are in an infinite semigroup, while it also tells how relatively small the minimal generating sets are.

(Lipcsey and Sampson, 2019) had discussed generating systems of semigroup with focus on dimension in which they utilized the isomorphism between the free semigroup over a one-letter alphabet $A = \{a\}$ and $(N^*, +)$, the additive semigroup of positive natural numbers, to construct examples for infinite, commutative and non-commutative semigroups, where there exists finite basis and the number of independent elements is larger than the basis. Their aim was to show (using the fact that every semigroup is isomorphic to a subsemigroup of a free semigroup), how the cardinalities of their generating sets related?

1.1. DEFINITION (SEMIGROUP)

Let S s be a set with a mapping $*: S \times S \in S$. Then S forms a semigroup $(S,*)$ if $\forall\, a, b, c \in S \Longrightarrow (a * b) * c = a * (b * c)$ holds.

1.2. DEFINITION (SUBSEMIGROUP)

Let $(S, *)$ be a semigroup. If a subset $H \subset S$ forms a semigroup for the same operation $*: H \times H \to H$ as in S, then H is a subsemigroup of S.

1.3. DEFINITION (MONOID)

Let $(S, *)$ be as semigroup. If there is a neutral element $e \in S$, such that $e * a = a * e = a$, $\forall a \in S$, then$S$is a Monoid.

1.4. LEMMA (Sampson, 2023).

Let $(S, *)$ be a semigroup. If S is not a monoid then there exists a set $M_S := S \cup \{e\}$and an operation $\star$ such that $e \star e = e$; $e \star a = a \star e = a$, $\forall\, a \in S$and $a \star b := a * b$, $\forall\, a, b \in S$.

Then $\{M_S, \star\}$is a monoid.

1.5. THEOREM.

Let $\{S,*\}$ be a semigroup. Let $H_\alpha \subset S$ be a subsemigroup$\forall \alpha \in A \neq \emptyset$, then

$$\emptyset \neq H = \bigcap_{\alpha \in A} H_\alpha$$

is a subsemigroup of S.

Proof.

Let $u, v \in H$. Then $u, v \in H_\alpha, \forall \alpha \in A \Longrightarrow$

$$u * v \in H_\alpha, \forall \alpha \in A \Longrightarrow u * v \in H = \bigcap_{\alpha \in A} H_\alpha$$

1.6. DEFINITION (COMMUTATIVE SEMIGROUP)

Let $(S, \cdot)$ be a semigroup. $(S, \cdot)$ is called a Commutative Semigroup, if

$$\forall x, y \in S : x \cdot y = y \cdot x .$$

1.7. DEFINITION (COSETS)

Let S be a semigroup containing the subsemigroup U, and let $c_i \in S$, then by the Left (right) Coset, $c_i U$ $(U c_i)$ of U, we mean the set of elements obtained by multiplying c_i on the right (left) by each element of U. A left coset of U that is also a right coset is simply called a Coset.

2.0. INFINITE SYSTEMS OF COSETS.

In section 2.1 of (Sampson & co, 2023) defined cosets of cyclic subsemigroups. Now we will define cosets of subsemigroups with special attention to subsemigroups of commutative (sub) semigroups as given by (Sampson, 2022).

2.1. DEFINITION (LEFT COSET AND RIGHT COSET)

Let $\{S,*\}$ be a semigroup and let $K \subset S$ be a subsemigroup.

Let $a \in S \backslash K$ then $a * K \subset S$ is a left coset and $K * a \subset S$ is a right coset of K.

2.2. LEMMA (LEFT AND RIGHT COSETS: COMMUTATIVE SEMIGROUP) (SAMPSON, 2022).

Let $C \subset S$ be a commutative subsemigroup and let $K \subset C \subset S$ be a subsemigroup of C. Then a left coset of K, $a * K \subset S$ and a right coset of $K, K * a \subset S$ are equal:

$$a * K = K * a, \ \forall a \in C \backslash K.$$

Proof. By commutativity of S, the products of elements of K by $a \in C \backslash K$, which can be written as $a * K$ or $K * a, \forall a \in C \backslash K$, would yield exactly the same value. So $a * K = K * a, \forall a \in C \backslash K$.

2.3. REMARKS.

The commutative subsemigroup$C$ in the lemma 2.2 will be mostly cyclic subsemigroup.

In (Effiong and co, 2023), semigroups with finite system of pairwise disjoint cosets have been considered. In this section we will give some results about an infinite system of cosets in a cyclic subsemigroup. We formulate an example in the form of a theorem below.

2.4. THEOREM (ON INFINITE CYCLIC FACTOR SEMIGROUP)

Let S be a semigroup and let $H \subset S$ be a cyclic subsemigroup

$$H := \langle \tau \rangle, s \in S.$$

Let $T \subset H$be a subsemigroup. Let $\tau^{\alpha} = \omega \in H \setminus T$. Then

$$\langle T \cup \{\omega\} \rangle = \bigcup_{k=0}^{\infty} \omega^k T, \quad \omega^0 = e \in S \qquad (5)$$

By the notations of the theorem 4.3,

$$\bigcup_{j=1}^{n} \langle N_j \rangle = \langle N \rangle = T$$

where each subsemigroup$\langle N_j \rangle, 1 \le j \le n$ is finitely generated except $j = n$ which may be infinite.

Therefore

$$\omega^k T = \omega^k \langle \{\tau^{a_t} | 1 \le t \le n\} \rangle.$$

If $n < \infty$,then

$$\omega^k T = \omega^k \{\tau^{\sum_{t=1}^{n} \omega_t n_t} | \, n_t \in \mathbb{N}, 1 \le t \le n\}$$

Using $a = s^{\alpha}$ and selecting $k = min(M_n) = a_1$, we get that

$$\omega^k * \tau^{\sum_{t=1}^{n} a_t n_t} = \tau^{\alpha\omega_1} \tau^{\sum_{t=1}^{n} \omega t n_t} = \tau^{\alpha * \omega_1 + \sum_{t=1}^{n} \omega_t n_t} = \tau^{r\omega_1(\alpha + n_1) + \sum_{t=2}^{n} \omega_t n_t} \in \langle N \rangle$$

Where

$$\alpha * k = \alpha * \omega_1 \text{ by } k = min(M_n) = \omega_1.$$

Thus $\omega^k T \subset T$

Therefore the factor a^k is the neutral elements $e$in the finite cyclic semigroup $\langle T \cup \{\omega\}\rangle / T$.

If $n$is infinity, then there is no power $k \in \mathbb{N}$such that $\omega^k * T \equiv T \, mod \, k$in the exponent.

Equivalently $a^k * T \subset T$.

Hence $\langle T \cup \{a\} \rangle / T$ is an infinite cyclic factor semigroup.

2.5. THEOREM (SAMPSON, 2022)

Let $\{S,*\}$be a semigroup. Let $H, C \subset S$, $H \cap C \neq \{\emptyset\}$ and $H \cap C \neq \{e\}$ be subsemigroups and let $C := \langle s \rangle$, $s \in S$ be a cyclic subsemigroup. Then $\emptyset \neq T := H \cap C$ fulfills the following:

1. $\forall \tau \in T$, $\exists! \, p \in N$ (there exists ***a unique*** *p in N*) such that $\tau = \alpha^p$.
2. $\forall (\emptyset \neq A) \subset T$, $\exists p_{min}(A) := min\{p \mid \alpha^p \in A\}$;

Proof. See Section 3.9.10 of (Sampson, 2022) for the proof)

2.6. IMPORTANT COROLLARY

(Sampson, 2022): Algorithm for Minimal Generating Set.

Let $S, H, C$and T be semigroups as defined in 2.5. Let M be initialized as $M \subset N$ and

Let $M = \emptyset$;Let$N \subset T$and let N be initialized as $N = \emptyset$;Let $\Gamma \subset C$ and let $\Gamma = T$. Let $n \in \mathbb{N}$ be the counter,

set to $n = 0$for starting. Then let us consider the following algorithm:

Step 1. If $\Gamma \neq \emptyset$; then goto Step 2 else [exit].

Step 2. Let $n = n + 1$, $k_n = p_{min}(\Gamma)$, $M = M \cup \{k_n\}$ and $N = N \cup \{s^{k_n}\}$goto step 3.

Step 3. $\Gamma = \Gamma \backslash \langle N \rangle$goto Step 1.

If this algorithm stops after finite steps then we get a set of powers which are independent

$$\forall p \in M, s^p \notin \langle \{s^t | t \in M, t < p\} \rangle$$

The value of$n$, $1 \leq n \leq \infty$gives the total number of minimums recorded.

Let a set $A \subset C$ be given. Then the ascending sequence M defined by the algorithm with $\Gamma = A$is unique.

Proof: See section 4.2 of (Sampson, 2022) for the proof of this theorem.

2.7. REMARK.

The theorem 2.4 are combined with the results 2.5 and 2.6 above by (Sampson, 2022) to give an example of generating set of a Semigroup below.

2.8. EXAMPLE (FINDING GENERATING SET IN $H\backslash T$).

We can apply the algorithm in the corollary 2.6 within $T \subset H \subset S$ for finding a generating set in $H\backslash T$. Precisely, let $M \subset \mathbb{N}$ and let $M = \emptyset$. Let $N \subset H$ and let $N = \emptyset$. Let $\Gamma \subset H$ and let $\Gamma = H\backslash T$.

Let $n \in \mathbb{N}$ be the counter, set to $n = 0$ for starting. This algorithm will select a co-prime (an independent) sequence of semi-ring elements such that $\langle T \cup N \rangle = C$ and the elements of N will be independent from T.

3.0. CONCLUSION AND RECOMMENDATION

With the above, basic constructions can be prepared for characterizing generating set and independent set in infinite semigroups. Also, what this shows is that independent elements are mostly infinite in an infinite semigroup, while the minimal generating sets are relatively small.

REFERENCES

Effiong, Louis Essien , Achuobi, Jonas Ogar , Eke, Nwagrabe (2023). On Cosets and Intersection of Cyclic Subsemigroups, International Journal of Science & Engineering Development Research (www.ijrti.org), ISSN:2455-2631, Vol.8, Issue 8, page no.151 - 155, 2023.

Howie, J. M., Marques Ribeiro, M. I: Rank properties in finite semigroups II. Southern Asian Bulletin of Mathematics, Springer-Verlag 2000.

Lipcsey, Z., & Sampson, M. I. (2019). *Rank of semigroup: generating sets and independent sets*. Proceedings of the 236th The IIER International Conference, 20th – 21st May, 2019, San Diego, Califonia, USA.

Lipcsey and Sampson (2023). On Existence of Minimal Generating Sets and Maximal Independent Sets in Groups and The Additive Semigroup of Integers. IOSR Journal of Mathematics (IOSR-JM) e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 19, Issue 4 Ser. 1, Page no. 57-64.

Sampson, M. I, Effiong L., Eke, N. (2023). Certain Equivalence in a Selected Family of Cyclic Subsemigroups

Sampson, M. I. (2022). Generating Systems of Semigroups and Independence. Ph.D. thesis, Department of Mathematics, University of Calabar, Calabar, Nigeria.

Sampson, M. I. (2023). Infinite Semigroups whose Number of Independent Elements is Larger than the Basis", International Journal of Science & Engineering Development Research (www.ijrti.org), ISSN:2455-2631, Vol.8, Issue 7, page no.1001 - 1005, http://www.ijrti.org/papers/IJRTI2307145.pdf

Tero, H. (1996). *Lecture notes on semigroups.*Department of Mathematics, University of Turku, FIN-20014 Turku, Finland.

