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# Abstract Approach of Functional Analysis

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## Abstract:

Functional analysis is an abstract branch of mathematics that originated from classical analysis. Its development started about eighty years ago, and nowadays functional analytic methods and results are important in various fields of mathematics and its applications. The impetus came from linear algebra, linear ordinary and partial differential equations, calculus of variations, approximation theory and, in particular, linear integral equations, whose theory had the greatest effect on the development and promotion of the modern ideas. Mathematicians observed that problems from different fields often enjoy related features and properties. This fact was used for an effective unifying approach towards such problems, the unification being obtained by the omission of unessential details. Hence the advantage of such an abstract approach is that it concentrates on the essential facts, so that these facts become clearly visible since the investigator's attention is not disturbed by unimportant details. In this respect the abstract method is the simplest and most economical method for treating mathematical systems. Since any such abstract system will, in general, have various concrete realizations (concrete models), we see that the abstract method is quite versatile in its application to concrete situations. It helps to free the problem from isolation and creates relations and transitions between fields which have at first no contact with one another.

**Key Words:** Functional, integral, abstract.

**Introduction and Concept:**

Functional analysis encompasses two primary interpretations: it serves as a mathematical discipline focused on the examination of vector spaces and their associated functions, and it functions as a methodological approach in psychology and engineering aimed at comprehending the purpose and operation of a system or behavior. In the realm of mathematics, it integrates linear algebra with analysis to address challenges in areas such as differential equations and quantum mechanics. In psychology, it is employed to investigate the underlying causes of behavior, while in engineering; it provides a framework for modeling a system based on its functions rather than its physical attributes.

The basic and historically first class of spaces studied in functional analysis is complete normed vector spaces over the real or complex numbers. Such spaces are called Banach spaces.

A vector space V over a field K (which covers C means complex numbers or R means real numbers) is a set of vectors which comes with an addition

$$+ : V \times V \to V$$

and scalar multiplication

$$\cdot : K \times V \to V,$$

along with some axioms: commutativity, associativity, identity, and inverse of addition, identity of multiplication, and distributivity.

**Example:** $R^n$ and $C^n$ are vector spaces, and so is C([0, 1]), the space of continuous functions [0, 1] → C. (This example is indeed a vector space because the sum of two continuous functions is continuous, and so is a scalar multiple of a continuous function.)

**Definition:** Let X be a linear (vector) space over the field K (which covers C means complex numbers or R means real numbers). A norm on X is a mapping (function) $\|\cdot\|$ from X to K, satisfying the following three axioms:

(1) $\| x \| \geq 0$ and $\| x\| = 0 \Rightarrow x = 0$[Positivity];

(2) $\| \lambda x \| = |\lambda| \| x \|$ for all $x \in X$ and all $\lambda \in K$ [Homogeneity]

(3) $\| x + y \| \leq \| x \| + \| y \|$ for all $x, y \in X$ [Triangle inequality]

X with the norm defined on it is called a normed space (or normed linear space).
A semi norm is a function $\|\cdot\|: V \to [0, \infty)$ which satisfies (2) and (3) but not necessarily (1), and a vector space equipped with a norm is called a normed space.

Let $\|\cdot\|$ be a norm on a vector space V.
Then

$$d(v, w) = \|v - w\|$$

defines a metric on V, which we call the "metric induced by the norm."

**Proof**: We just need to check the three conditions above: property (1) of the norm implies property (1) of metrics, because

$$d(v, w) = \|v - w\| = 0$$
$$\Leftrightarrow v - w = 0$$
$$\Leftrightarrow v = w.$$

For property (2) of the metric, note that

$$\|v - w\| = \|(-1)(w - v)\| = |-1| \cdot \|w - v\| = \|w - v\|,$$

by using property (2) of the norm.
And finally, property (3) of the metric is implied by property (3) of the norm because

$$(x - y) + (y - z) = (x - z)$$

**Example:** The Euclidean norm on $R^n$ or $C^n$, given by

$$\|x\|_2 = \left( \sum_{i=0}^{n} |\mathbf{x}|^2\right)^{1/2},$$

is indeed a norm (this is the standard notion of "distance" that we're used to). But we can also define

$$\|x\|_\infty = \max_{1\leq i\leq n} | x_i |$$

(the "length" of a vector is the largest magnitude of any component), and more generally (for $1 \leq p < \infty$)

$$\|x\|_p = \left( \sum_{i=0}^{n} |\mathbf{x}|^{\mathbf{p}}\right)^{1/p},$$

We can illustrate the concept of "unit balls" in $R^2$ for the various norms. It is important to remember that B(x, r) represents the collection of points that are at a maximum distance of r from x: under the norm $\|\cdot\|_2$, B(0, 1) appears as a circle, whereas under the norm $\|\cdot\|_\infty$, B(0, 1) resembles a square with vertices located at (±1, ±1). Additionally, under the norm $\|\cdot\|_1$, it takes the form of a square with vertices at (0, 1), (1, 0), (0, −1), and (−1, 0). Generally speaking, the various $\|\cdot\|_p$ norms will produce "unit balls" that fall between the two squares described above.

So changing the norm does change the geometry of the balls, but not too drastically: if we take a large enough $L^1$ ball (that is, a ball B (0, r) with large enough r under the $\|\cdot\|_1$ norm), it will always swallow up an $L_\infty$ ball of any fixed size. This "sandwiching" basically means that the norms are essentially equivalent.

**Definition:** Let X be a metric space. The vector space $C_\infty(X)$ is defined as

$$C_\infty(X) = \{f : X \to C : f \text{ continuous and bounded}\}.$$

For example, $C_\infty([0, 1])$ is $C([0, 1])$, because all continuous functions on [0, 1] are bounded.

**Proposition**: For any metric space X, we can define a norm on the vector space $C_\infty(X)$ such as

$$\|u\|_\infty = \sup_{x\in X} |u(x)|.$$

**Proof:** Properties (1) and (2) of a norm are clear from the definitions, and we can show property (3) as follows. If $u, v \in C_\infty(X)$, then for any $x \in X$, we have

$$|u(x) + v(x)| \leq |u(x)| + |v(x)|$$

by the triangle inequality for C, and this is at most $\|u\| + \|v\|$ (because u(x) is bounded by its supremum, and so is v (x)). Thus, we indeed have

$$|u(x) + v(x)| \leq \|u\|_\infty + \|v\|_\infty \quad \forall x \in X$$
$$\Rightarrow \|u + v\|_\infty = \sup x\, |u(x) + v(x)| \leq \|u\|_\infty + \|v\|_\infty$$

And now that we have a norm, we can think about convergence in that norm:

we have $u_n \to u$ in $C_\infty(X)$ (convergence of the sequence)

$$\text{if } \lim_{n\to\infty} \|u_n - u\|_\infty = 0,$$

which we can unpack in more familiar analysis terms as $\forall \varepsilon > 0, \exists\, n \in N$ :

$$\forall n \geq N, \forall x \in X, |u_n(x) - u(x)| < \varepsilon,$$

which is the definition of uniform convergence on X. Thus, convergence in this metric (we will use the terms metric and norm interchangeably, as the metric is derived from the norm) essentially represents a condition of uniform convergence when dealing with bounded, continuous functions.

**Definition:** A normed space is a Banach space if it is complete with respect to the metric induced by the norm.

**Theorem:** For any metric space X, the space of bounded, continuous functions on X is complete, and thus $C_\infty(X)$ is a Banach space.

**Proof:** We want to show that every Cauchy sequence $\{u_n\}$ converges, meaning that it has some limit u in $C_\infty(X)$. This proof basically illustrates how we prove that spaces are Banach in general: take a Cauchy sequence, come up with a candidate for the limit, and show that (1) this candidate is in the space and (2) convergence does occur. So if we have our Cauchy sequence $\{u_n\}$, first we show that it is bounded under the norm $C_\infty(X)$. To see this, note that there exists some positive integer $N_0$ such that for all $n, m \geq N_0$,

$$\|u_n - u_m\|_\infty < 1$$

So now for all $n \geq N_0$,

$$\|u_n\|_\infty \leq \|u_n - u_{N0}\|_\infty + \|u_{N0}\|_\infty < 1 + \|u_{N0}\|_\infty$$

by the triangle inequality, and thus for all $n \in N$, we have

$$\|u_n\|_\infty \leq \|u_1\|_\infty + \cdots + \|u_{N0}\|_\infty + 1$$

(because we need to make sure the first few terms are also small enough). So we can bound $\|u_n\|_\infty$ by some finite positive B, and thus we have a bounded sequence in the space $C_\infty(X)$. So now if we focus on a particular $x \in X$, we have

$$|u_n(x) - u_m(x)| \leq \sup x\, |u_n(x) - u_m(x)|$$
$$= \|u_n - u_m\|_\infty,$$

and because $\{u_n\}$ is Cauchy, for any $x \in X$, the sequence of complex numbers $\{u_n(x)\}$ (where we evaluate each function un at the fixed x) is a Cauchy sequence. But the space of complex numbers is a complete metric space, so for all $x \in X$, $u_n(x)$ converges to some limit, which will help us define our candidate function:

$$u(x) = \lim_{n\to\infty} u_n(x).$$

This is basically the point wise limit, and we now need to show this is in $C_\infty(X)$ and that we have convergence under the uniform convergence norm. Now we know that

$$|u(x)| = \lim_{n\to\infty} |u_n(x)|$$

(if the limit exists, so does the limit of the absolute values), and now we know that the right-hand side is bounded by the $\|\cdot\|_\infty$ norm, and thus by the statement that we found above. That means that $\sup x \in X\, |u(x)| \leq B$, so u is indeed a bounded function. To finish the proof, we'll show continuity and convergence, which we'll do with the usual definition. Fix $\varepsilon > 0$; since $\{u_n\}$ is Cauchy, there exists some N such that for all $n, m \geq N$, we have

$$\|u_n - u_m\|_\infty < \varepsilon/2$$

So now for any $x \in X$, we have

$$|u_n(x) - u_m(x)| \leq \|u_n - u_m\|_\infty < \varepsilon/2$$

so taking the limit as $m \to \infty$, we have that for all $n \geq N$,

$$|u_n(x) - u(x)| \leq \varepsilon/2$$

(every thing is still point wise at a point x here). So it's also true that

$$\sup x = |u_n(x) - u(x)| \leq \varepsilon/2 < \varepsilon,$$

and thus

$$\|u_n - u\|_\infty \to 0$$

And now because $\|u_n - u\|_\infty \to 0$, we know that $u_n \to u$ uniformly on X, and the uniform limit of a sequence of continuous functions is continuous.

  

Therefore, our candidate u is in $C_\infty(X)$ and is the limit of the $u_n$, and thus $C_\infty(X)$ is complete and a Banach space.

**Conclusion:**

Normed spaces hold significant importance as they establish a foundational framework in both mathematics and machine learning for quantifying the "size" or "length" of vectors, which facilitates the understanding of distance and convergence. This aspect is vital for the analysis and optimization of algorithms, given that the selection of a norm can greatly influence model performance in areas such as regression. They integrate the algebraic characteristics of vector spaces with the topological properties of metric spaces, thereby forming the cornerstone for disciplines like functional analysis.

Banach spaces hold significant importance as they offer a robust and precise framework for examining issues in infinite-dimensional spaces, especially in domains such as functional analysis, differential equations, and optimization. The property of completeness guarantees that sequences that are expected to converge indeed do converge, which is vital for resolving equations and establishing theorems. They serve as indispensable instruments across various scientific disciplines for the formulation and analysis of intricate systems.

**References:**

- Conway, John (1990). A course in functional analysis. Graduate Texts in Mathematics. Vol. 96 (2nd ed.). New York: Springer-Verlag, ISBN 978-0-387-97245-9. OCLC 21195908
- Conway, John (1990). A course in functional analysis. Graduate Texts in Mathematics. Vol. 96 (2nd ed.). New York: Springer-Verlag. ISBN 978-0-387-97245-9. OCLC 21195908
- Megginson, Robert E. (1998), An introduction to Banach space theory, Graduate Texts in Mathematics, vol. 183, New York: Springer-Verlag, pp. xx+596, ISBN 0-387-98431-3.
- Rudin, Walter (1991). Functional Analysis. International Series in Pure and Applied Mathematics. Vol. 8 (Second ed.). New York, NY: McGraw-Hill Science/Engineering/Math. ISBN 978-0-07-054236-5. OCLC 2116327
- Rolewicz, Stefan (1987), Functional analysis and control theory: Linear systems, Mathematics and its Applications (East European Series), vol. 29 (Translated from the Polish by Ewa Bednarczuk ed.), Dordrecht; Warsaw: D.Reidel Publishing Co.; PWN—Polish Scientific Publishers, pp. xvi+524, doi:10.1007/978-94-015-7758-8, ISBN 90-277-2186-6,

  