Variational symmetries and Conservation Laws,for Soliton equations in (2+1) Dimentions

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Abstract

In this paper,we study Variational symmetries and Conservation Laws ,and we obtain a direct relationship between Variational symmetries and Conservation Laws of the potential modified Korteweg-de Vries and the sin -Gorden equations , K.P.equation, and equaion of higher order

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1 Introduction

To find conservation laws we showed that it is fruitful if one examines . Consider Euclidean space , with $x = R^p$, with coordinate $x = (x^1, ...x^p)$ representing the independent variables , and $U = R^q$ with coordinate $u = (u^1, ..., u^q$ the dependent variables , let $\Omega \subset x$ be an open , connected subset with smooth boundary $\delta\Omega$, a Variational problem consists of a functional transformations which leave the action integral

$$J[u] = \int_\Omega L(x, u^n)dx$$

Invariant [1, 8,9] where L is called Euler -Lagrangian equation . we establish a direct relationship between invariances and conservation laws we obtain a direct relationship between Variational symmetries and Conservation Laws of the K-dV equations and the S-G equation , Boussinesq equation , K.P. equaion following the recent works on higher order is seen as follows:
Consider a partial equation of the form

Definition 1.1 *(Variational symmetries)*

A transformation

$$x^* = x$$

$$u* = u + \varepsilon\eta(x, u, u_1, ...u_p + 0\varepsilon^2)$$

is a Variational symmetries of integral $J[u]$ *, if for any* $u(x)$ *there exists some vector function*

$$A(x, u, u_1, .., u_r)$$

of x *,* u *and its derivatives to some finite order* r*, sucht that if* u^k *is the infinitesimal generator[2] of a variational symmetry of an action integral*

$$U^k L = D_i A^i \tag{1}$$

Where

$$U^k = \eta^\nu \frac{\delta}{\delta u^\nu} + \eta_i^\nu \frac{\delta}{\delta u_{i_1}^\nu} + ... + \eta_{i_1 i_2 .. i_k} \frac{\delta}{\delta u_{i_1 i_2 ... i_k}}$$

And

$$D_i = \frac{\delta}{\delta x_i} + u_i \frac{\delta}{\delta u} + u_{i_j} \frac{\delta}{\delta u_{i_j}} + ...$$

Holds for any u(x) , then [1,2] the conservation law

$$D_i[(W_i[u, \eta]) - A^i] = 0 \tag{2}$$

Holds for any solution $u(x)$ of Euler- Lagrange equation $E_y(L) = 0$,$\nu = 1, ..., m$ where

$$W^i[u, \eta] = \eta\nu[\frac{\delta L}{\delta u_\nu} - D_{i_1} \frac{\delta}{\delta u_{i_1}^\nu} + (-1)^{k-1} D_{i_1} D_{i_{k-1}} \frac{\delta L}{\delta u_{i....i_{k-1}}}]$$

2 Examples

We give some examples to make use of this formulism and obtain conservation law for some Soliton equations.

Exampl (1) the Sin-Gorden equation

let Sin- Goden equaion [4,5] is

$$u_{1t} - sinu = 0$$

Consider

$$L = -\frac{1}{2} u_1 u_t + cosu$$

and

$$\eta = [u_{111} + \frac{1}{2}u_1^3]$$

The corresponding Euler - Lagrange equation to Sin - Gorden equation here is

$$U^2L = -\frac{1}{2}D_t(u_1\eta) + \frac{1}{2}u_{1t}\eta - \frac{1}{2}D_1(u_1\eta) + \frac{1}{2}u_{1t}\eta - sinu\eta$$

$$U^2L = -\frac{1}{2}u_1[u_{111t} + \frac{3}{2}u_1^2u_{1t}] - \frac{1}{2}u_t[u_{1111} + \frac{3}{2}u_1^2u_{11}]$$

$$-sinu[u_{111} + \frac{1}{2}u_1^3] \tag{3}$$

Form (1)$U^2L = D_1A^1 + D_2A^2$ where
A^1 and A^2 are vector functions of (x, u) and its derivatives
then from (3)

$$A^1 = \{-\frac{1}{2}u_t(u_{111} + u_1^3) - u_{11}u_{1t} - u_{11}sinu + \frac{1}{2}cosu\}$$

$$A^2 = \{-\frac{1}{2}u_1u_{111} - \frac{3}{8}u_1^4 + \frac{1}{2}u_{11}^2\}$$

from (2)

$$U^2L = -\frac{1}{2}u_1D_t\eta - \frac{1}{2}D_{1\eta} - sinu\eta$$

Consequently

$$U^2L = Eu\eta + D_1W^1[u,\eta] + D_tW^2[u,\eta]$$

$$EL(\eta) = \frac{1}{2}U_{1t} - sinu$$

$$= u_{1t} - sinu$$

and

$$W^1 = -\frac{1}{2}u_t\eta, W^2 = -\frac{1}{2}u_1\eta$$

Now applying the conservation law (2)
first

$$D_1(W^1 - A^1)$$

$$= [u_{111}u_{1t} - u_{111}sinu + u_1u_{11}cosu - u_1u_{11}cosu - \frac{1}{2}u_1^2sinu] \tag{4}$$

And

$$D_t(W^2 - A^2) = D_t[-\frac{1}{2}u_{11}^2 + \frac{1}{8}u_1^4] - u_1u_{11t} + \frac{1}{2}u_1^3u_{1t} \tag{5}$$

Then from (4) , (5) the conservation law becomes

$$[u_{111} + \frac{1}{2}u_1^3][u_{1t} - sinu] = 0$$

Example (2) K.d.V equation
Let K. d .V equation[3,4] is

$$V_{xxx} + VV_x + V_t = 0$$

Let

$$V = u_x = u_1$$

Then

$$u_{xxxx} + u_x u_{xx} + u_{xt} = 0$$

Then the transformed is

$$u_{1111} + u_1 u_{11} + u_{1t} = 0$$

Which is the Euler -Lagrange equation for

$$L = \frac{1}{2}u_{11}^2 - \frac{1}{6}u_1^3 - \frac{1}{2}u_1 u_2$$

Let U^2 be twice extended operator of

$$U = \eta \frac{\delta}{\delta u}$$

Where

$$U^2 L = EL\eta + D_1 W^1[u_1\eta] + D_t W^2[u_t\eta]$$

then

$$U^2 L = u_{11} D_1^2 \eta - \frac{1}{2}u_1^2 D_1 \eta - \frac{1}{2}u_t D_1 \eta$$

consequently , where

$$EL = u_{1111} + u_1 u_{11} + u_{1t}$$

$$W^1[u, \eta] = u_{11} D_1 \eta - u_{111}\eta - \frac{1}{2}u_1^2 \eta - \frac{1}{2}u_t \eta$$

$$W^2[u, \eta] = \frac{1}{2}(u, \eta)$$

and determined

$$A^1 = t[\frac{1}{2}u_{11}^2 - \frac{1}{6}u_1^3 - \frac{1}{2}u_1 u_t]$$

$$A^2 \frac{1}{2}u$$

$$\eta = (tu_1 - x)$$

Then we obtain the conservation law

$$D_1(W^1 - A^1) + D_t(W^2 - A^2) = 0$$

wich we can calculate as :

$$D_1(W^1 - A^1) = D_1[u_{11} D_1(tu_1 - x) - u_{111}(tu_1 - x) - \frac{1}{2}u_1^2(tu_1 - x) - \frac{1}{2}u_t(tu_1 - x)]$$

$$-D_1 t[\frac{1}{2}u_{11}^2 - \frac{1}{6}u_1^3 - \frac{1}{2}u_1 u_t]$$

$$= D_1[tu_{11}^2 - u_{11} - tu_{111}u_1 + xu_{111} - \frac{1}{2}tu_1^3 + \frac{1}{2}xu_1^2 - tu_1u_t + \frac{1}{2}xu_t - \frac{1}{2}tu_{11}^2 + \frac{1}{6}tu_1^3 + \frac{1}{2}tu_1u_t]$$

$$= D_1[t(-\frac{1}{2}u_{11}^2 - \frac{1}{3} - u_1u_{111}) - u_{11} + xu_{111} + \frac{1}{2}x(u_1^2 + u_2)]$$

D is total derivative respect to x

$$D_t(W^1 - A^1) = (tu_{11}u_{111} - tu_1^2u_{11} - tu_1u_{111})$$

$$-tu_{11}u_{111} - u_{111} + u_{111} + xu_1u_{11} + \frac{1}{2}u_t + \frac{1}{2}xu_{1t} + xu_{1111} + \frac{1}{2}u_t^2 \quad (6)$$

And D_t is defined analogously.Without loss of generality, we assume that η does depend on t- derivatives since for admissible operators can always eliminated using equation (1)
Then

$$D_t(W^2 - A^2) = D_t(-\frac{1}{2}u_1(tu_1 - x) - \frac{1}{2}u)$$

$$= D_t[-\frac{1}{2}tu_1^2 + \frac{1}{2}xu_1 - \frac{1}{2}u]$$

$$= [-\frac{1}{2}u_1^2 - tu_1u_{1t} - \frac{1}{2}u_t] \quad (7)$$

From (6) and (7) we obtain the conservation law , which here become

$$D_1(W^1 - A^1) + D_t(W^2 - A^2) = 0$$

$$= -tu_1(u_{1111} + u_1u_{11} + u_{1t}) + x(u_{1111} + u_1u_{11} + u_{1t}) = 0$$

3 conservation laws for Boussinesq equation

We illustrate here equaion of higher order to obtain higher order[6,7]
conservation laws for Boussinesq equation
let

$$u_{tt} + 2uu_{xx} + 2u_x^2 + u_{xxxx} = 0$$

The transformed Boussinesq equation is

$$u_{1111} + 2uu_{11} + 2u_1^2 + u_{tt} = 0$$

Wich is Euler - Lagrange equation

$$L = \frac{1}{2}u_{11}^2 - uu_1^2 - \frac{1}{2}u_t^2$$

Let U^2 be twice extended operator then

$$U^2L = u_{11}D_1^2\eta - 2uu_1D_1\eta - u_tD_t\eta$$

Where

$$U^2 L = EL\eta + D_1 W^1[u,\eta] + D_t W^2[u,\eta]$$

Where

$$EL = u_{1111} + 2u_1^2 + 2uu_{11} + u_{1t}$$
$$W^1[u,\eta] = u_{11}D_1\eta + u_{111}\eta - 2uu_1\eta$$
$$W^2[u,\eta] = -[u_t,\eta]$$

Then we obtain the coservation law ( ) Where

$$D_1(W^1 - A^1) = [-tu_1^3 + tu_tu_{1t} - tu_1u_{1111} + xu_{1111} - 2tuu_1u_{11} + 2xu_1^2 + xu_1u_{11}] \quad (8)$$

$$D_t(W^2 - A^2) = D_t[-u_t[tu_1 - x] - u] \quad (9)$$

Thus from (8) and (9) the coservation law:

$$D_1(W^1 - A^1) + D_t(W^2 - A^2) = -tu_1[u_{1111} + 2u_1^2 + 2uu_{11} + u_{tt}]$$
$$+x[u_{1111} + 2u_1^2 + 2uu_{11} + u_{tt}] = 0$$

Where

$$[u_{1111} + 2u_1^2 + 2uu_{11} + u_{tt}] = 0$$

4 in the 3- dimensional case of K.P. equation

We take one of higher dimensional equations,[4 ,5] called the Kadomtsev - Petviashvili K . P. equation to get the relation between variational symmetry and coservation law which is written as

$$D_1(W^1 - A^1) + D_t(W^2 - A^2) - D_y(W^3 - A^3) = 0 \quad (10)$$

in the 3- dimensional case of K.P. equation
Consider K.P. equation as

$$(u_{xxx} + \frac{1}{2}u_x^2)_x + \sigma u_{yy} = 0$$

Where σ is a constant , $\sigma = \pm 1$
Then the transformed $K.P.$ equation is

$$u_{1111} + u_1u_{11} + u_{1t} + \sigma u_y^2$$

is the Euler - Lagrange equation for Lagrangian L

$$L = \frac{1}{2}u_{11}^2 - \frac{1}{6}u_1^3 - \frac{1}{2}u_1u_t - \frac{1}{2}\sigma u_y^2$$

Let U^2 be twice extended operator of $U = \eta\frac{\delta}{\delta u}$ Then

$$U^2 L = u_{11}D_1^2\eta - \frac{1}{2}u_1^2 D_1\eta - \frac{1}{2}u_1 D_t\eta - \frac{1}{2}u_t D_1\eta - \sigma u_y D_y\eta$$

where

$$U^2L = EL\eta + D_1W^1[u,\eta] + D_tW^2[u,\eta] + D_yW^3[u,\eta]$$

Where

$$EL = u_{1111} + u_1u_{11} + u_{1t} + \sigma u_{yy} = 0$$

$$W^1[u,\eta] = u_{11}D_1\eta - u_{111}\eta - \frac{1}{2}\eta[u_1^2 + u_t]$$

$$W^2[u,\eta] = -\frac{1}{2}[u_1\eta]$$

$$W^3[u,\eta] = -[\sigma u_y\eta]$$

And determined

$$A^1 = t[\frac{1}{2}u_{11}^2 - \frac{1}{6}u_1^3 - \frac{1}{2}u_1u_t - \frac{1}{2}\sigma u_y^2]$$

$$A^2 = \frac{1}{2}u$$

$$A^3 = \sigma u$$

$$\eta = (tu - x - y)$$

Then we obtain the coservation law which becomes

$$D_1(w^1 - A^1) = D_1[u_{11}D_1(tu_1 - x - y) - u_{111}(tu_1 - x - y) - \frac{1}{2}u_1^2(tu_1 - x - y) - \frac{1}{2}u_t(tu_1 - x - y)]$$

$$-D_1t[\frac{1}{2}u_{11}^2 - \frac{1}{6}u_1^3 - \frac{1}{2}u_1u_t - \sigma u_y^2] \quad (11)$$

$$D_t[W_2 - A^2] = D_t[\frac{1}{2}u_1(tu_1 - x - y) - \frac{1}{2}u] \quad (12)$$

$$D_y[W^3 - A^3] = D_y[-\sigma u_y(tu_1 - x - y) - \sigma u] \quad (13)$$

from (11) , (12) ,(13)we obtain the conservation law which here become

$$D_1(W^1 - A^1) + D_t(W^2 - A^2) - D_y(W^3 - A^3)$$
$$= xu_{1111} + yu_{1111} - tu_1^2u_{11} + xu_1u_{11}$$
$$-tu_1u_{1111} + yu_{1t} + yu_1u_{11} + \frac{1}{2}xu_{1t}$$
$$+\frac{1}{2}yu_{1t} - tu_1u_{1t} + \frac{1}{2}xu_{1t} + \frac{1}{2}yu_{1t} - \sigma tu_1u_{yy} + \sigma xu_{yy} + \sigma yu_{yy}$$

So we get

$$u_{1111} + u_1u_{11} + u_1t + \sigma u_{yy} = 0$$

And the conservation law is (10)

$$D_1(W^1 - A^1) + D_t(W^2 - A^2) - D_y(W^3 - A^3)$$
$$= -tu_1[u_{1111} + u_1u_{11} + u_{1t} + \sigma u_{yy}]$$
$$+x[u_{1111} + u_1u_{11} + u_{1t} + \sigma u_{yy}]$$
$$+y[u_{1111} + u_1u_{11} + u_{1t} + \sigma u_{yy}] = 0$$

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