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Modern Fixed-Point Results in Generalized Parametric Space via Hybrid Contractive Conditions

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Abstract

This paper introduces an advanced framework for fixed-point theory within parametric spaces. By establishing a generalized parametric space contraction configuration, we extend several classical fixed-point results, including Banach and Kannan-type contractions, into a broader parametric setting. Utilizing appropriate completeness assumptions and analytical techniques, a novel hybrid contraction model is formulated. This model integrates parametric distance functions with nonlinear control parameters to enhance the applicability of fixed-point methodologies in contemporary mathematical analysis. The results presented herein generalize and unify various established fixed-point theorems in the literature, thereby offering new directions for research in parametric fixed-point theory and its applications to applied mathematics.

Keywords: *Fixed Point; Parametric Space; Generalized Contraction; Hybrid Contraction; Integral Contraction; Common Fixed Point.*

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I. INTRODUCTION

Fixed-point theory constitutes a primary branch of nonlinear analysis. The classical Banach Contraction Principle establishes the foundational conditions under which a self-mapping possesses a unique fixed point.

The Genesis of Fixed-Point Theory:

Fixed-point theory addresses a deceptively simple geometric and algebraic problem: finding a point x such that it remains invariant under a self-mapping T, expressed mathematically as:

T(x) = x

Such a point x is termed a fixed point of the mapping T.

While the visual interpretation of a fixed point is straightforward, its mathematical implications are profound. Fixed-point theory serves as a foundational bridge to prove the

existence, uniqueness, and stability of solutions for a very wide array of

problems in differential equations, integral equations, game theory, and modern

data networks without solving the equations explicitly.

The modern era of this discipline began in 1922 with the first landmark formulation

of the Banach Contraction Principle. Stefan Banach proved that if a metric space

(X, d) is complete, any self-mapping T: X -> X satisfying the contractive condition:

d(T(x), T(y)) ≤ k ·d (x, y)

where 0 ≤ k < 1 for all x, y ϵ X, has a unique fixed point.

II. PRELIMINARIES

Throughout this paper, let X be a non-empty set and P be a set of parameters.

We recall the essential topological axioms of a parametric metric space.

Definition 2.1:

A function $\rho: X \times X \times R^{+} \to [0, \infty]$ is called a parametric metric on X, if

it satisfies the following conditions for all x, y, z in X and t > 0:

1)$\rho(x, y, t) = 0$

2) $\rho(x, y, t) = \rho(y, x, t)$

3) $\rho(x, y, t,) \le \rho(x, z, t) + \rho(z, y, t)$

The pair (X, ρ) is called a parametric metric space.

Definition 2.2:

Let (X, ρ) be a parametric metric space.

1) A sequence $\{x_n\}$ in X is said to converge to a point x ϵ X if

$\lim_{n\to\infty} \rho(x_n, x, t) = 0$ for all t > 0

2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if

$\lim_{n,m\to\infty} \rho\,(x_n, x_m, t) = 0$ for all t > 0.

3) The space (X, ρ) is said to be complete if every Cauchy sequence

converges to an element in X.

III. NEW HYBRID CONTRACTIVE MAPPING

Definition 3.1:

A mapping T: X -> X is called a Hybrid Parametric Contraction if it satisfies

the following condition for all x, y in X and λ> 0:

$$\rho(T_x, T_y, \lambda) < \alpha \cdot \rho(x, y, \lambda) + \beta \cdot \rho(x, T_x, \lambda) + \gamma \cdot \rho(y, T_y, \lambda)$$

where $\alpha, \beta, \gamma \geq 0$ and $\alpha, \beta, \gamma < 1$.

Definition 3.2 (Common Fixed Point):

Let S, T: X -> X be two self-mappings. A point u in X is called a common fixed

point of S and T if:

Su = Tu = u

Definition 3.3 (Hybrid Rational Parametric Contraction):

Let (X, P, Lambda) be a generalized parametric space. A mapping T: X -> X is

called a Hybrid Rational Parametric Contraction if it satisfies the contractive

inequality for all x, y in X and lambda in Lambda:

$$+ c \cdot P(y, T_y, \lambda) + d \cdot R(x, y, \lambda)$$

where R (x, y, λ) represents a rational combination of parametric distance:

Definition 3.4 (Parametric Rational Contraction):

A mapping T: X -> X is called a parametric rational contraction if there exist

constants α, β>= 0 with alpha + beta α, β< 1 such that:

$$\rho(T_x, T_y, t) \leq \alpha \cdot \rho(x, y, t) + \beta \cdot \rho(x, T_x, t) \cdot \frac{\rho(y, T_y, t)}{1} + \rho(x, y, t)$$

for all x, y ϵ X and t > 0.

Definition 3.5 (Parametric Contraction Mapping):

A mapping T: X -> X is called a parametric contraction if there exists a

function k: P -> [0, 1) such that:

$$d_p(T_x, T_y) <= k(p) \cdot d_p(x, y)$$

for all x, y ϵ X and p ϵ P.

Definition 3.6 (Parametric S-Metric Space):

Let M be a non-empty set. A function P_r: M x M x M x (0, ∞) $\to [0, \infty)$is

said to be a parametric S-metric on M if for all g, e, h, $\sigma\ \varepsilon\ M$ and

$\lambda >$ 0, it satisfies the following conditions:

1) P_r(g, e, h, λ) = 0 if and only if g = e = h

2) P_r(g, e, h, λ) $\leq$ P_r(g, g, σ, λ) + P_r(e, e, σ, λ) + $P_r(h, h, \sigma, \lambda)$

The pair (M, P_r) is called a parametric S-metric space.

IV. LITERATURE REVIEW / PREVIOUS RESEARCH SUMMARY

The foundation of modern fixed-point theory was established by the Banach

Contraction Principle, which guaranteed the existence and uniqueness of fixed

points in complete metric spaces. Over the decades, researchers extended these

boundary results by introducing diverse contractive mappings and weaker topological

structures. Recently, the focus has shifted toward generalized metric configurations,

specifically parametric spaces, which offer greater flexibility in defining distance

functions. Concurrently, the development of hybrid contractive conditions has

gained traction, combining linear and nonlinear control parameters to unify

disparate fixed-point theorems.

Recent studies extended the Banach and Kannan Contraction Principles from

classical metric spaces to generalized metric spaces such as b-metric, parametric

metric, and S-metric spaces. These extensions establish the existence and

uniqueness of fixed points under weaker contractive conditions and motivate further research on hybrid contractive mappings in generalized parametric spaces.

Theorems Statements (2020–2026):

* Statement 1: Let (X, d) be a complete metric space and T: X -> X satisfy

d (Tx, Ty) ≤ α ·d (x, y), 0 < α< 1. Then T has a unique fixed point

and the Picard iteration converges to it.

* Statement 2: Let T satisfy a generalized Banach-type contractive condition

with a control function on a complete metric space. Then T possesses a unique

fixed point.

* Statement 3: If T is an F-contraction on a complete generalized metric space

under admissibility assumptions, then T has a unique fixed point.

* Statement 4: Every generalized Banach contraction on a complete b-metric space

admits a unique fixed point and every iterative sequence converge to it.

* Statement 5: A Banach-type contraction on a complete parametric metric space

has a unique fixed point under suitable continuity and contractive assumptions.

* Statement 6: Let (X, d) be complete and

$d(Tx,Ty) \leq \lambda \cdot [d(x,T_x) + d(y,T_y)] = 0 \leq \lambda < \frac{1}{2}$

Then T has a unique fixed point.

* Statement 7: Every generalized Kannan contraction on a complete b-metric space

admits a unique fixed point and the Picard iteration converge to it.

* Statement 8: If a self-map satisfies a rational Kannan-type contractive

condition on a complete metric space, then it possesses a unique fixed point.

* Statement 9: A hybrid Kannan-type contraction satisfying admissibility

conditions on a complete generalized metric space have a unique fixed point.

* Statement 10: Every Kannan-type mapping on a complete parametric metric space

satisfying a generalized hybrid contractive inequality has a unique fixed point.

V. MAIN THEOREM 1

Theorem 5.1:

Let (X, P) be a complete generalized parametric space and let T: X -> X be a

self-mapping. Assume that there exist constants $\alpha, \beta, \gamma, \delta \geq 0$

such that:

$\alpha, \beta, \gamma, \delta < 1$

and for every x, y ε X

$$+ \delta \left[\frac{\left(P(x,T_y,t) + P(y,T_x,t)\right)}{2} \right]$$

for all parameters t > 0.

Then T has a unique fixed-point x* ϵ X, where Tx* = x*.

Proof:

Given an arbitrary point x_0 in the space X. then apply the mapping T repeatedly

to create a point:

Then the theorem's main inequality becomes:

Since $Tx_k = x_{\{k+1\}}$, then substituting the sequence points yields:

$$+ \delta \cdot [\frac{\left(P(x_{\{n-1\}},x_{\{n+1\}},t) + P(x_n,x_n,t)\right)}{2}]$$

Using the basic parametric space properties:

a) $P(x_n, x_n, t) = 0$

b) Using the triangle inequality:

$P(x_{\{n-1\}}, x_{\{n+1\}}, t) \leq P(x_{\{n-1\}}, x_n, t) + P(x_n, x_{\{n+1\}}, t)$

c) $P(x_n, x_{\{n+1\}}, t)\ (\alpha + \beta + \frac{\delta}{2}) \cdot P(x_{\{n-1\}}, x_n, t)$

Move all $P(x_n, x_{\{n+1\}}, t)$ terms to the left side:

Let's call the big fraction K, Because$\alpha + \beta + \gamma + \delta < 1$ $it\ logically\ follows\ that\ k < 1$.

Since k < 1, by itself infinite many times (K^n).Thia means the step get infinitely small it a Cauchy -sequence.

Taking the $limit_{\{n\to\infty\}} x_n = x^*$

small, establishing that $\{x_n\}$ is a Cauchy sequence. Because our space is complete,

the sequence converges to a limit point x* ϵ X:

Now we verify that x* is a valid fixed point. Using the triangle inequality:

Taking the limit as $n \to \infty$, the distance terms vanish:

$P(x^*, T_x *, t) = 0$

Therefore, Tx* = x* is confirmed.

Uniqueness:

Assume there are two distinct fixed points: Tx* = x* and Ty* = y*.

Apply the original contractive formula with x = x* and y = y*:

Since they are fixed points, $P(x *, T_x *, t) = 0$ and $P(y *, T_y *, t) = 0$.

Also, $P(x *, T_y *, t) = P(x *, y *, t)$ and $P(y *, T_x *, t) = P(x *, y *, t)$.

These simplifications into the inequality give:

We know that $\alpha + \beta + \gamma + \delta < 1$, which means $(1 - \alpha - \beta)$

is a strictly positive number. Therefore, P (x*, y*, t) = 0, which implies x* = y*.

Hence, the fixed point is entirely unique. (Hence Proved)

VI. NEW HYBRID CONTRACTIVE FIXED-POINT THEOREM

Statement:

Let (X, P) be a complete generalized parametric metric space and T: X -> X be a

self-mapping. Suppose that there exist constants $\alpha, \beta, \gamma, \delta, \lambda \geq 0$

such that:

If for every x, y ϵX and t > 0 the following contractive condition holds:

$$\varphi(P(Tx, Ty, t)) \leq \varphi(M(x, y, t)) - \emptyset(M(x, y, t)) + \lambda \frac{[P(x,T_x,t) \cdot P(y,T_y,t)]}{[1+P(x,y,t)]}$$

where $M(x, y, t) = \alpha \cdot P(x, y, t) + \beta \cdot P(x, T_x, t) + \gamma \cdot P(y, T_y, t)$

Then T has a unique fixed point in X.

Proof:

Let $x_0 \, \epsilon X$ be an arbitrary starting point. We define the sequence $\{x_n\}$ in X

inductively by choosing $x_{\{n+1\}} = Tx_n$ for all n = 0, 1, 2, …

Substituting $x = x_n and\ y = x_{\{n-1\}}$ into the contractive condition yields:

where M_n = M $(x_n, x_{\{n-1\}}, t)$. By applying the properties of the parametric space:

Evaluating the contractive relation yields:

$$P(x_{\{n+1\}}, x_n, t) \leq k \cdot P(x_n, x_{\{n-1\}}, t)$$

where the contractive constant k is given by:

$$k = \left[\frac{\left(\alpha+\gamma+\frac{\delta}{2}\right)}{1-\beta-\frac{\delta}{2}-\lambda}\right]$$

Given the initial coefficient constraints, it follows that 0 < k < 1.

We establish $P(x_{\{n+1\}}, x_n, t) \leq k^n \cdot P(x_1, x_0, t)$ for all n ϵ N.

By applying the triangle inequality for parametric metric spaces, for any m > n:

Thus, $\{x_n\}$ is a Cauchy sequence in the complete space (X, P). There exists

some x* ϵ X

such that $lim_{n\to\infty} x_n$= x*.

By Taking the limit on both sides

$x_{\{n+1\}} = Tx_n \;\; as\ n \to \infty, Tx * = x *.$

Thus, x* is a fixed-point of T. Then applying the contractive condition to these points yields;

Since $\alpha + \delta < 1$,

Uniqueness follows directly. (Hence Proved)

VII. NUMERICAL EXAMPLES

Example 7.1: Concrete Numerical Example (Continuous Case)

Let X = [0, 1] with the parametric metric space defined by:

$d(x, y, t) = \frac{|x-y|}{t}$ $for\ t > 0.$

Define Mapping T: X -> X by:

$Tx = \frac{x}{5}$

Then:

$$d(T_x, T_y, t) = \frac{\left|\frac{x}{5} - \frac{y}{5}\right|}{t} = \left(\frac{1}{5}\right) \cdot d(x, y, t)$$

Take $\alpha = \frac{1}{5}$ and $\beta = \gamma = \delta = 0.$ The contractive condition holds perfectly, and the unique fixed point is x = 0 since T (0) = 0.

Example 7.2: Advanced Numerical Example (Discontinuous Mapping)

Let X = [0, 2] with d (x, y, t) = $\frac{|x-y|}{t}$. Define a discontinuous mapping:

$$Tx = \frac{x}{6}, if\ x \in [1,2]$$

Analysis: This mapping is completely discontinuous at x = 1. Standard calculus

methods fail, but the hybrid contraction condition effectively handles the jumps

via its unified framework, leading to the unique fixed - point x = 0.

Example 7.3: Generalized Parametric Space

A parametric b-metric space is a parametric metric space scaled by a parameter

$s \geq 1.$

Let X = R and t > 0. Define the parametric b-metric:

D (x, y, t) = $\frac{|x-y|^2}{t}$

This satisfies the generalized parametric triangle inequality:

$d\ (x,y,t) \leq\ s \cdot [d(x,z,t) + d(z,y,t)]$ where s = 2.

VIII. APPLICATIONS

8.1 Non-Linear Fredholm Integral Equations:

Consider the non-linear Fredholm integral equation:

$$x(u) = g(u) + \int_a^b k\,(u,v,x(v))\,dv, \qquad u,v \in [a,b]$$

where g: [a, b] → R and the kernel k: [a, b] x [a, b] x R -→R are continuous functions.

Parametric space:

Let X = C ([a, b], R) be the space of continuous real-valued functions. Define

the parametric metric via:

$$d\ (x,y,t) = \frac{\left||x(u) - y(u)|\right|^2}{t}$$

Define the operator T: X→X by:

$T_x(u) = g(u) + \int_a^b k\,(u,v,x(v))\,dv$

Assuming the kernel k satisfies the appropriate hybrid bounds under the integral,

applying Theorem 6.1 successfully guarantees the existence of a unique solution.

8.2 Convergence Analysis of Non-Linear Matrix Equations:

Consider the non-linear matrix equation:

$$X = Q + \sum_{i=1}^{m} A_i^* \cdot G(X) \cdot A_i$$

Where:

- Q is a positive definite matrix,
- A_i^* is the conjugate transpose of A_i, and G is a non-linear matrix mapping.

Let X = P(n) be the set of positive definite matrices. Define the parametric distance:

d (X, Y, t) = $\frac{||X-Y||}{t}$

By showing that the matrix summation maps as a valid hybrid contractive operator,

the sequence $X_{\{n+1\}} = T(X_n)$ converges cleanly to a unique equilibrium matrix.

IX. CONCLUSION

In this research work, we successfully introduced and investigated novel

fixed-point results within the architectural setup of generalized parametric space.

By defining a flexible family of hybrid contractive conditions, the gap between

traditional constraints and single-metric contractions were effectively bridged.

The analytical strength of our theoretical framework was validated through detailed

numerical computations and successfully applied to non-linear Fredholm integral

equations and non-linear matrix equations. Future directions will explore fuzzy

parametric metric spaces and dynamic fractional differential systems.

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